MINIMUM MEAN SQUARE ERROR FILTERING OVER THE CLASS OF EXTENDED THRESHOLD BOOLEAN FILTERS

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ABSTRACT

Recently, a class of nonlinear digital filters, called the threshold Boolean filter (TBF), is introduced. The TBF is defined by a Boolean function on the binary domain and is a natural extension of stack filters. In this paper, the TBF is further extended to a larger class of filters, called the extended TBF (ETBF), which encompasses linear FIR and linear combination of order statistic (LOS) filters as well as the TBF. The optimal design of the TBF and ETBF under the mean square error (MSE) criterion is investigated. It is shown that the optimization of the TBF can be formulated as a quadratic zero-one programming, and that the optimization of the ETBF as a classical quadratic problem. Thus, linear FIR and median-type nonlinear filters can be analyzed and designed in a unified framework of the ETBF. The ETBF has been applied to enhance noisy images. The results show that the ETBF outperforms the TBF and the FIR Wiener filter.

1. INTRODUCTION

In [1], the threshold Boolean filter (TBF) has been introduced as an extension of stack filters, where the stacking (positivity) constraint on a Boolean function is removed, and multi-level representations of the TBF have been derived. In this paper, the class of TBFs is further extended by employing an extended binary-input function which can produce real valued outputs for binary input vectors. This extension results in a very large class of filters, what is called the *extended threshold Boolean filter* (ETBF), which encompasses linear FIR and LOS[2] filters as well as the TBF. Thus the structure of the ETBF provides a unified theoretical tool for analysis and design of both linear and nonlinear filters.

In Section 2, definitions of the stack filter, TBF and ETBF are presented. The relation between the ETBF to linear FIR and LOS filters is investigated in Section 3. The problem of designing an optimal TBF and ETBF under the mean square error (MSE) criterion is considered in Section 4. It is shown that the optimal design of TBFs can be formulated into a *quadratic zero-one programming*, and that the optimization of the ETBF becomes a classical quadratic problem. An adaptive filtering procedure for ETBFs, based on LMS algorithm[3], is addressed in Section 5. Results from computer simulations on performance characteristics of the proposed filter is presented in Section 6. Finally, Section 7 gives conclusions.

2. STACK, THRESHOLD BOOLEAN, AND EXTENDED THRESHOLD BOOLEAN FILTERS

A nonrecursive digital filter is a mapping (an operator) $\beta : \mathcal{D} \mapsto \mathcal{V}$, where $\mathcal{D}$ is the domain set on which input signals into the filter are defined and $\mathcal{V}$ is the range set from which the output of the filter takes its values. Consider an input vector $\mathbf{X}(k) = (X_1(k), \cdots, X_N(k))$ consisting of input values within a window. Here $X_i(k)$ is the i-th sample within a window located at k and N is the size of the moving window. We assume that each signal sample $X_j(k)$ takes on some value in the set $\mathcal{Q} = \{0, 1, \cdots, M\}$; that is, the signal has been quantized into $M+1$ levels. Thus, $\mathcal{D} = \mathcal{Q}^N$ and the range set $\mathcal{V}$ is assumed to be either $\mathcal{Q}$ or $\mathcal{R}$, the set of real values. Henceforth, the location (time) index k will be dropped to simplify notation. Thresholding a sample X_j at a level ℓ, denoted by $T^\ell(X_j)$, is a mapping from $\mathcal{Q}$ to $\mathcal{B} = \{0, 1\}$, defined as $T^\ell(X_j) = 1$ if $X_j \geq \ell$ and 0, otherwise. Obviously the following relationship holds: $X_j = \sum_{\ell=1}^{M} T^\ell(X_j)$ for $j = 1, \cdots, N$. By thresholding, any signal vector $\mathbf{X} \in \mathcal{D}$ can be decomposed into M binary vectors $\mathbf{x}^\ell = (x_1^\ell, \cdots, x_N^\ell) \in \mathcal{B}^N$, $\ell = 1, \cdots, M$ where $x_j^\ell = T^\ell(X_j)$, $j = 1, \cdots, N$.

Definition 1 *A filter $\beta : \mathcal{D} \mapsto \mathcal{V}$ is said to obey the threshold decomposition if for any input $\mathbf{X} \in \mathcal{D}$, β satisfies $\beta(\mathbf{X}) = \beta\left(\sum_{\ell=1}^{M} T^\ell(\mathbf{X})\right) = \sum_{\ell=1}^{M} \beta\left(T^\ell(\mathbf{X})\right) = \sum_{\ell=1}^{M} f_\beta(\mathbf{x}^\ell)$ where $f_\beta : \mathcal{B}^N \mapsto \mathcal{V}$ is a binary-input operator associated with β.*

If a filter obeys the threshold decomposition, the output of the filter for a given input vector can be obtained by decomposing the input vector into the set of M binary vectors, carrying out the associated binary filtering separately on each binary vectors, and then by summing up the binary outputs. Note that a filter obeying the threshold decomposition can be specified by the associated binary operator, instead of its multi-level operator. A binary operator $f : \mathcal{B}^N \mapsto \mathcal{B}$ is a Boolean function of N variables.

Definition 2 *A binary operator $f : \mathcal{B}^N \mapsto \mathcal{B}$ is said to possess the stacking property if $\mathbf{b}_1 \leq \mathbf{b}_2$ implies $f(\mathbf{b}_1) \leq f(\mathbf{b}_2)$ for any two binary vectors $\mathbf{b}_1, \mathbf{b}_2 \in \mathcal{B}^N$. Here $\mathbf{b}_1 \leq \mathbf{b}_2$ if each element of $\mathbf{b}_2$ is greater than or equal to its corresponding element of $\mathbf{b}_1$.*

0-7803-1775-0/94 $3.00

It is known that a binary operator possesses the stacking property if and only if it can be expressed as a positive Boolean function[4].

2.1. Stack Filters

Based on the threshold decomposition and the stacking property, stack filters[5] are defined as follows:

Definition 3 *For an input vector* $\mathbf{X} \in \mathcal{D}$*, the output of the stack filter based on a positive Boolean function* $f(\cdot)$*, denoted by* $S_f(\mathbf{X})$*, is defined as*

$$S_f(\mathbf{X}) = \sum_{\ell=1}^{M} f(\mathbf{x}^\ell). \tag{1}$$

The multi-level operation $S_f(\cdot)$ corresponding to a positive Boolean function $f(\cdot)$ is obtained via (1) as a composition of *min* and *max* operations[6].

2.2. Threshold Boolean Filters

A natural extension of the class of stack filters is obtained by removing the stacking constraint on $f(\cdot)$ in (1), i.e., by allowing $f(\cdot)$ to be an arbitrary Boolean function. The resultant new filter class is called the threshold Boolean filter (TBF)[1]. For a given window size N, there is one-to-one correspondence between TBFs and Boolean functions.

Definition 4 *For an input vector* $\mathbf{X} \in \mathcal{D}$*, the output of a TBF based on a Boolean function* $f(\cdot)$*, denoted by* $TBF_f(\mathbf{X})$*, is defined as*

$$TBF_f(\mathbf{X}) = \sum_{\ell=1}^{M} f(\mathbf{x}^\ell). \tag{2}$$

The multi-level TBF expressions corresponding to an arbitrary Boolean function have been derived in [1]. Among them, the optimization procedure exploits the following representation theorem:

Proposition 1 *Let* $\mathbf{v}_j = (v_{j1}, \cdots, v_{jN})$ *be the radix-2 representation of an integer* $j \in \{0, 1, \cdots, 2^N - 1\}$*. Then, the output of a TBF can be expressed as*

$$TBF_f(\mathbf{X}) = \mathbf{a}^t \mathbf{W} \tag{3}$$

where $\mathbf{a} = [a_0 \ \ a_1 \ \ \cdots \ \ a_{2^N-1}]^t$ *and* $\mathbf{W} = [W(\mathbf{X}|\mathbf{v}_0) \ W(\mathbf{X}|\mathbf{v}_1) \cdots W(\mathbf{X}|\mathbf{v}_{2^N-1})]^t$ *are column vectors with* $a_i = f(\mathbf{v}_i)$ *and* $W(\mathbf{X}|\mathbf{v}_i) = max\{0, min\{X_j \mid v_{ij} = 1, \ j = 1, \cdots, N\} - max\{X_j \mid v_{ij} = 0, \ j = 1, \cdots, N\}\}$*, and min and max of the empty set are defined to be M and 0, respectively.*

2.3. Extended Threshold Boolean Filters

We extend the TBF by replacing the Boolean function $f(\cdot)$ in (2) with an extended binary-input function $g : \mathcal{B}^N \mapsto \mathcal{R}$. The resulting filter class is called the *extended TBF* (ETBF).

Definition 5 *For an input vector* $\mathbf{X} \in \mathcal{D}$*, the output of an ETBF based on a binary-input function* $g(\cdot)$*, denoted by* $ETBF_g(\mathbf{X})$*, is defined as*

$$ETBF_g(\mathbf{X}) = \sum_{\ell=1}^{M} g(\mathbf{x}^\ell). \tag{4}$$

Any binary-input function $g(\cdot)$ can be specified by its output vector $\mathbf{c} = [c_0 \ \ c_1 \ \ \cdots \ \ c_{2^N-1}]^t$ where $c_j = g(\mathbf{v}_j) \in \mathcal{R}$, $j = 0, 1, \cdots, 2^N - 1$. Apparently, the ETBF includes the TBF as a special case when $c_j \in \mathcal{B}$ for all $j = 0, 1, \cdots, 2^N - 1$. By extending (3), $ETBF_g(\mathbf{X})$ can be written as a linear combination of $W(\mathbf{X}|\mathbf{v}_i)$'s, as shown below.

Proposition 2 *A multi-level representation of the ETBF is*

$$\begin{aligned} ETBF_g(\mathbf{X}) &= \sum_{j=0}^{2^N-1} c_j \, W(\mathbf{X}|\mathbf{v}_j) \\ &= \mathbf{c}^t \mathbf{W}. \end{aligned} \tag{5}$$

The ETBF class encompasses a large variety of filters including TBF, linear FIR and LOS filters. This relation will be examined in the next section.

3. RELATING ETBFS TO LINEAR FIR AND LOS FILTERS

In this section, we investigate what constraints on $\mathbf{c}$ in (5) force the ETBF to fall into some existing filter classes.

The output of a linear FIR filter with N taps is given by $Y = \sum_{j=1}^{N} h_j X_j$ where $h_j \in \mathcal{R}$, $j = 1, \cdots, N$ are filter coefficients. Again, let $\mathbf{v}_i = (v_{i1}, \cdots, v_{iN})$ be the radix-2 representation of an integer $i \in \{0, 1, \cdots, 2^N - 1\}$, where v_{i1} is the MSB and v_{iN} is the LSB.

Proposition 3 *An ETBF is reduced to a linear FIR filter, i.e.,*

$$ETBF_g(\mathbf{X}) = \sum_{j=1}^{N} h_j X_j$$

if $c_i = \sum_{j=1}^{N} v_{ij} h_j$*,* $i = 0, 1, \cdots, 2^N - 1$.

Proof : It follows directly from the definition of the ETBF: $ETBF_g(\mathbf{X}) = \sum_{\ell=1}^{M} g(\mathbf{x}^\ell) = \sum_{\ell=1}^{M} \left(\sum_{j=1}^{N} x_j^\ell h_j\right) = \sum_{j=1}^{N} h_j \left(\sum_{\ell=1}^{M} x_j^\ell\right) = \sum_{j=1}^{N} h_j X_j$.

Example 1 *When* $N = 3$*,* $ETBF_g(\mathbf{X}) = \sum_{j=1}^{3} h_j X_j$ *if* $c_0 = 0$*,* $c_1 = h_3$*,* $c_2 = h_2$*,* $c_3 = h_2 + h_3$*,* $c_4 = h_1$*,* $c_5 = h_1 + h_3$*,* $c_6 = h_1 + h_2$*, and* $c_7 = h_1 + h_2 + h_3$.

The output of an LOS filter (originally called order statistic filter in [2] and sometimes called the L-filter) with window width N is given by $Y = \sum_{j=1}^{N} h_j X_{(j)}$ where $h_j \in \mathcal{R}$, $j = 1, \cdots, N$ are filter coefficients and $X_{(j)}$ is the j-th largest sample within the window: $X_{(1)} \geq \cdots \geq X_{(N)}$. Let $w_H(\mathbf{v}_i)$ denote the Hamming weight of $\mathbf{v}_i$; $w_H(\mathbf{v}_i) = \sum_{j=1}^{N} v_{ij}$. Then, the following proposition states the condition for the ETBF to be an LOS filter.

Proposition 4 *An ETBF is reduced to an LOS filter, i.e.,*

$$ETBF_g(\mathbf{X}) = \sum_{j=1}^{N} h_j X_{(j)}$$

if $c_0 = 0$ *and* $c_i = \sum_{j=1}^{w_H(\mathbf{v}_i)} h_j$*,* $i = 1, \cdots, 2^N - 1$.

Proof : Suppose $\mathbf{x}^\ell \neq \mathbf{0}$. Then, $1 \leq w_H(\mathbf{x}^\ell) \leq N$ and $T^\ell(X_{(j)}) = 1$, if $1 \leq j \leq w_H(\mathbf{x}^\ell)$ and 0, otherwise. Hence, we derive the identity $g(\mathbf{x}^\ell) = \sum_{j=1}^{w_H(\mathbf{x}^\ell)} h_j = \sum_{j=1}^{N} T^\ell(X_{(j)})h_j$. Since $g(\mathbf{0}) = 0$ and $T^\ell(X_{(j)}) = 0$ for all j if $\mathbf{x}^\ell = \mathbf{0}$, the identity still holds for $\mathbf{x}^\ell = \mathbf{0}$. From the definition of the ETBF, $ETBF_g(\mathbf{X}) = \sum_{\ell=1}^{M} g(\mathbf{x}^\ell) = \sum_{\ell=1}^{M}\sum_{j=1}^{N} T^\ell(X_{(j)})h_j = \sum_{j=1}^{N} h_j \left(\sum_{\ell=1}^{M} T^\ell(X_{(j)})\right) = \sum_{j=1}^{N} h_j X_{(j)}$.

Example 2 *When* $N = 3$, $ETBF_g(\mathbf{X}) = \sum_{j=1}^{3} h_j X_{(j)}$ *if* $c_0 = 0$, $c_1 = c_2 = c_4 = h_1$, $c_3 = c_5 = c_6 = h_1 + h_2$, *and* $c_7 = h_1 + h_2 + h_3$.

4. MINIMUM MEAN SQUARE ERROR FILTERING

Suppose that $\mathbf{s}(k)$ and $\mathbf{r}(k)$ are desired and observed signals, respectively, where $\mathbf{r}(k)$ is a distorted version of $\mathbf{s}(k)$. It is assumed that $\mathbf{s}(k), \mathbf{r}(k) \in \mathcal{Q}$ for all k. The observation at k within a moving window is denoted by a vector $\mathbf{X}(k) = (X_1(k), \cdots, X_N(k)) = (\mathbf{r}(k-L_1), \cdots, \mathbf{r}(k), \cdots, \mathbf{r}(k+L_2))$ where the window size is $N = L_1 + L_2 + 1$ with nonnegative integers L_1 and L_2. From this observation, we wish to find an operator $\beta(\cdot)$ which minimizes the mean square error (MSE), $MSE(\beta) = E\{|\mathbf{s}(k) - \beta(\mathbf{X}(k))|^2\}$. In this study, $\beta(\cdot)$ is confined to be either a stack filter or a TBF, or an ETBF. Therefore, the $MSE(\beta)$ can be rewritten as

$$MSE(\beta) = MSE(\mathbf{a}) = E\{|\mathbf{s} - \mathbf{a}^t\mathbf{W}|^2\} \tag{6}$$

where $\mathbf{a} = [a_0 \quad a_1 \quad \cdots \quad a_{2^N-1}]^t$ is a column vector with

$$a_j \in \begin{cases} \mathcal{B}, & \text{for } \beta(\cdot) \text{ to be a stack filter or TBF} \\ \mathcal{R}, & \text{for } \beta(\cdot) \text{ to be an ETBF} \end{cases}$$

for $j = 0, 1, \cdots, 2^N - 1$. By expanding the right hand side of (6), we get

$$\begin{aligned} MSE(\mathbf{a}) &= E\{|\mathbf{s} - \mathbf{a}^t\mathbf{W}|^2\} \\ &= E\{\mathbf{s}^2\} - 2\mathbf{a}^t E\{\mathbf{s}\mathbf{W}\} + \mathbf{a}^t E\{\mathbf{W}\mathbf{W}^t\}\mathbf{a} \\ &= \xi - 2\mathbf{a}^t\Phi + \mathbf{a}^t\Psi\mathbf{a}, \end{aligned} \tag{7}$$

where $\xi = E\{\mathbf{s}^2\}$, $\Phi = E\{\mathbf{s}\mathbf{W}\}$, $\Psi = E\{\mathbf{W}\mathbf{W}^t\}$. Note that Φ is a 2^N vector and Ψ is a $2^N \times 2^N$ matrix. The entries of Ψ are all nonnegative because $W(\mathbf{X}|\mathbf{v}_j) \geq 0$ for all j. Furthermore, Ψ is symmetric, i.e., $\psi_{ij} = \psi_{ji}$. If $\mathbf{v}_i$ and $\mathbf{v}_j$ are incomparable, $\psi_{ij} = \psi_{ji} = 0$ since both $W(\mathbf{X}|\mathbf{v}_i)$ and $W(\mathbf{X}|\mathbf{v}_j)$ cannot have simultaneously nonzero values for a nonzero input vector $\mathbf{X}$.

Assume that Φ and Ψ are known; in most practical cases, however, they should be estimated from known $\mathbf{s}(k)$ and $\mathbf{r}(k)$. Apparently, the minimizer $\mathbf{a}^*$ of (7) also minimizes $J(\mathbf{a}) = \frac{1}{2}\mathbf{a}^t\Psi\mathbf{a} - \mathbf{a}^t\Phi$ and *vice versa*. Therefore, we have the following optimization problem:

Optimization 1

$$\underset{\mathbf{a}}{\textit{minimize}} \;\; J(\mathbf{a}) = \frac{1}{2}\mathbf{a}^t\Psi\mathbf{a} - \mathbf{a}^t\Phi$$

In the following, we recast Optimization 1 for stack filters, TBFs, and ETBFs.

4.1. Optimal Stack Filtering

By the definition of stack filters (Definition 3), the entries of the column vector $\mathbf{a}$ in Optimization 1 should be the outputs of a positive Boolean function. Thus, a_j's should be binary and obey the stacking constraint, $a_i \geq a_j$ whenever $\mathbf{v}_i \geq \mathbf{v}_j$, which results in

Optimization 2

$$\begin{gathered} \underset{\mathbf{a}}{\textit{minimize}} \;\; J(\mathbf{a}) = \tfrac{1}{2}\mathbf{a}^t\Psi\mathbf{a} - \mathbf{a}^t\Phi \\ \textit{subject to} \;\; a_i \geq a_j \text{ whenever } \mathbf{v}_i \geq \mathbf{v}_j \\ a_j \in \mathcal{B} \;\; \forall\, j \end{gathered}$$

This is a constrained *quadratic zero-one programming* problem, and unfortunately no practically useful algorithm to solve this has been found at this time.

4.2. Optimal Threshold Boolean Filtering

Since the TBF is free from the stacking constraint, Optimization 2 is simplified to

Optimization 3

$$\begin{gathered} \underset{\mathbf{a}}{\textit{minimize}} \;\; J(\mathbf{a}) = \tfrac{1}{2}\mathbf{a}^t\Psi\mathbf{a} - \mathbf{a}^t\Phi \\ a_j \in \mathcal{B} \;\; \forall\, j \end{gathered}$$

This is a quadratic zero-one programming known as an *NP-hard* problem[7]. Some numerical search and heuristic techniques to solve this problem can be found in the literatures[8]-[10].

4.3. Optimal Extended Threshold Boolean Filtering

For an extended TBF, the entries of $\mathbf{a}$ are not necessarily binary but real values. Consequently, Optimization 1 becomes a classical quadratic problem:

Optimization 4

$$\begin{gathered} \underset{\mathbf{a}}{\textit{minimize}} \;\; J(\mathbf{a}) = \tfrac{1}{2}\mathbf{a}^t\Psi\mathbf{a} - \mathbf{a}^t\Phi \\ a_j \in \mathcal{R} \end{gathered}$$

This formulation is routinely encountered in the optimization of linear filters. By differentiating $J(\mathbf{a})$ with respect to $\mathbf{a}$ and setting it to zero, we get the closed form solution $\mathbf{a}^* = \Psi^{-1}\Phi$. The inversion of an $n \times n$ matrix takes in general $O(n^3)$ arithmetic operations, and it takes $O(2^{3N})$ operations to calculate Ψ^{-1}. We next consider an adaptive procedure for designing an ETBF, which does not require the inversion of Ψ.

5. ADAPTIVE FILTERING

In practice, the statistics required to obtain Ψ and Φ are usually unknown. Moreover, the size of matrix Ψ increases exponentially as a function of N. In order to overcome these difficulties, we consider an adaptive procedure for designing ETBFs. Since Optimization 4 is to minimize a quadratic

Table 1. The MSE values between the original and filtered images.

	MSE		
	Gaussian	impulsive	Gaussian+impulsive
identity	91.99	1848.06	1185.13
Wiener	60.06	386.80	304.77
TBF	90.64	113.62	145.08
ETBF	57.87	96.74	126.62

function, the LMS algorithm[3] can be applied directly. Accordingly, the adaptive procedure for designing ETBFs is written as

$$e(k) = s(k) - \mathbf{a}^t(k)\mathbf{W}(k) \quad (8)$$

$$\mathbf{a}(k+1) = \mathbf{a}(k) + 2\mu\, e(k)\mathbf{W}(k) \quad (9)$$

where $e(k)$ is the error signal at time k and μ is a step size. Both $\mathbf{a}$ and $\mathbf{W}$ have 2^N entries, respectively. For a given input $\mathbf{X}$ at any time k, however, $\mathbf{W}(k)$ can have at most $N+1$ non-zero entries corresponding to the binary vectors $T^{X_{(i)}}(\mathbf{X})$, $i = 0, 1, \cdots, N$. Thus, both the calculation of $e(k)$ in (8) and updating $\mathbf{a}(k)$ in (9) have $O(N)$ computational complexity, which is comparable to the linear case. However, its adaptation period should be considerably longer than the linear case since the number of coefficients to be adapted for an ETBF is 2^N.

6. APPLICATIONS TO IMAGE PROCESSING

In this section, the Wiener filter, the TBF, and the ETBF are applied to suppress noise superimposed on an image, and their performance characteristics are compared.

We used the 512 × 512 "bridge" image as the original image, and synthesized from it three distinctive images corrupted with additive Gaussian, impulsive, and both Gaussian and impulsive noise, respectively. The generated Gaussian noise has mean zero and variance $\sigma_n^2 = 100$, and impulses occur with probability $P_e = 0.1$ and height ± 200. For each noisy image, correlation matrices required for determining the optimal coefficients of Wiener filter, TBF and ETBF were estimated from the upper-left quarter of the original and noisy images. In this experiment, 3 × 3 square window was used, i.e., $N = 9$. The optimal zero-one coefficients of the TBF were obtained by using the *depth-first branch and bound algorithm* proposed in [9]. Then, the designed filters were applied to the entire region of the corresponding noisy image for noise suppression. Their filtering performances have been compared in the MSE sense, and the resulting MSE values are listed in Table 1. For all types of noise, the ETBF performs best. As expected, the Wiener filter performs very close to the ETBF for Gaussian noise, but performs poorly when impulsive noise is present. The TBF works well under impulsive noise environments, but can do almost nothing against Gaussian noise. The results show that the ETBF can efficiently suppress both Gaussian and impulsive noise.

7. CONCLUSIONS

The TBF has been extended to a larger class of filters, which is called the extended TBF (ETBF). It has been shown that the ETBF includes not only median-type nonlinear filters such as the median filter, stack filters and TBFs but also linear FIR and LOS filters. Based on the multi-level representation theorem for the ETBF, the design of an optimal ETBF under the MSE criterion has been formulated as a classical quadratic problem. Computer simulations showed that the ETBF outperforms the TBF and the Wiener filter.

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