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THEORETICAL ANALYSIS OF WINSORIZING SMOOTHERS AND THEIR APPLICATIONS TO IMAGE PROCESSING

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ABSTRACT

The Winsorizing smoother (W smoother), which is a center weighted median (CWM) filter giving more weight only to the central value of each window, is studied. This filter can preserve image details while suppressing additive white and/or impulsive-type noise. The statistical properties of the W smoother are analyzed. It is shown that the W smoother can outperform the median filter, while its implementation is almost as simple as median filtering. Some relationships between W smoothers and other median-type filters, such as the weighted median filter and the multi-stage median filter, are derived.

Introduction

The median filter is a simple nonlinear smoothing operation which takes a median value of the data inside a moving window of finite length. This filter has been recognized as a useful image enhancement technique due to its edge preserving smoothing characteristics and its simplicity in implementation [1]. Median filtering preserves edges in images and is particularly effective in suppressing impulsive noise. Application of median filtering to an image, however, requires some caution because median filtering tends to remove image details such as thin lines and corners while reducing noise. Recently, in response to these difficulties, several variations of median filters have been introduced. Specifically, the max/median [2], FIR-median hybrid [3],[4] and multi-stage median [4],[5] filters have been developed for detail-preserving smoothing. These variations of median filters preserve more image details at the expense of noise suppression.

In this paper we investigate the *Winsorizing* smoother which was proposed in [6]. This smoother allows a degree of control of the smoothing behavior through a parameter which can be set, and thus, is a promising image enhancement technique. It will be shown that this smoother is identical to the CWM filter [7]-[9] which gives more weight only to the central value of a window, and thus is easier to design and implement than general weighted median filters [10], [11]. We shall analyze the properties of the W smoothers and observe that the W smoothers preserve more details at the expense of less noise suppression.

II. Winsorizing Smoothers

Before defining the W smoother, we review the median filter and introduce some common terminology. If we let $\{X(.,.)\}$ and $\{Y(.,.)\}$ be the input and output respectively, of the median filter, then

$$Y(i,j) = median\{X(i-s, j-t) \mid (s,t) \in R\}. \qquad (1)$$

Here R is the window which is defined in terms of the image coordinates in the neighborhood of the origin. For example, $(2N+1)$ x $(2N+1)$ square window is given by $R = \{(s,t) \mid -N \leq s \leq N, -N \leq t \leq N\}$. The total number of points in a window is called the window size. Throughout, the window size is denoted by $2L+1$.

Consider a W(2L+1,a) smoother where $2L+1$ is the window size and a is the filter parameter that controls the degree of smoothing. The output $Y(i,j)$ of the W(2L+1, a) smoother is defined by

$$Y(i,j) = \begin{cases} X_{ij}(a; 2L+1), & \text{if } X(i,j) \leq X_{ij}(a; 2L+1) \\ X_{ij}(2L+2-a; 2L+1), & \text{if } X(i,j) \geq X_{ij}(2L+2-a; 2L+1) \\ X(i,j), & \text{otherwise} \end{cases} \qquad (2)$$

where $X_{ij}(r; 2L+1)$ is the r^{th} smallest one among $2L+1$ samples within the window centered at (i,j), $X(i,j)$ is the input value at the center of the window, and $1 \leq a \leq L+1$.

The definition of the W smoother in (2) can be expressed as follows:

Definition: The output $Y(i,j)$ of the W(2L+1,a) smoother is represented by

$$Y(i,j) = median\{X_{ij}(a; 2L+1), X_{ij}(2L+2-a; 2L+1), X(i,j)\} \qquad (3)$$

When $a = L+1$, the W smoother becomes the median filter, and when $a = 1$, it becomes the identity filter (no filtering). Obviously, a W smoother with a smaller a performs better in detail preservation but worse in noise suppression than one with a larger a.

The W-smoother is very simple to implement. The values $X_{ij}(a; 2L+1)$ and $X_{ij}(2L+2-a; 2L+1)$ in (3) can be obtained efficiently by using algorithms which are slight modifications of the fast algorithms for median filtering in [12], [13]. When such algorithms are applied, the average number of comparisons for the W smoother with a 2-D square window is $O(\sqrt{L+1})$.

There is an interesting relationship between the W smoother and the CWM filter. The following property shows that the W(2L+1,a) smoother is identical to the CWM filter with central weight 2L+3-2a.

Property 1: The output $Y(i,j)$ of the W(2L+1,a) smoother is equivalent to

$$Y(i,j) = median\{X(i-s, j-t),\ 2L+2-2a \text{ copies of } X(i,j) \mid (s,t) \in R\} \qquad (4)$$

Consider the W(3,1) smoother with $R = \{(-1,0),(0,0),(1,0)\}$. Its output is equivalent to the output of the CWM filter having the central weight equal to three that is $Y(i,j) = median\{X(i-1,j), X(i,j), X(i,j), X(i,j), X(i+1,j)\}$. Therefore, the properties of W smoothers discussed in the paper are in fact those of

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CWM filters.

The W smoother also has an interesting relationship with a multi-stage median filter [4], [5], which employs several 1-D median filters, and either, the central sample in the window or, one of the filter outputs is selected using a decision logic based on a median filtering algorithm.

Property 2: The multi-stage median filter (unidirectional) with a 3x3 square window is identical to the W(9,2) smoother with the 3x3 square window.

This equivalence between the W smoothers and the multi-stage median filters does not hold in general if the window size is greater than 3x3; see [8] for an example.

III. Statistical Properties of W Smoothers

In this section the noise suppression, edge and detail preservation characteristics of W smoothers are statistically studied. The noise suppression characteristics of W smoothers are investigated when a constant signal is embedded in additive white noise. The edge and detail preservation properties are examined by considering 2-D inputs with step edges and lines which are also corrupted by additive white noise.

A. Noise Suppression

The probability distribution function for the output of the W smoother is given in the next property.

Property 3: For independently and identically distributed (i.i.d.) inputs, the output distribution function $F_{\mathbf{Y}}(y)$ of the W(2L+1,a) smoother is given by

$$F_{\mathbf{Y}}(y) = \sum_{j=k_1-1}^{2L} \binom{2L}{j} F_{\mathbf{X}}^{j+1}(y)(1 - F_{\mathbf{X}}(y))^{2L-j} + \sum_{j=k_2}^{2L} \binom{2L}{j} F_{\mathbf{X}}^{j}(y)(1 - F_{\mathbf{X}}(y))^{2L+1-j} \quad (5)$$

where $k_1 = a$, $k_2 = 2L+2-a$, and $F_{\mathbf{X}}(.)$ is the input distribution. It may be noted that if $F_{\mathbf{X}}(x)$ is symmetric about m then $F_{\mathbf{Y}}(y)$ in (5) is also symmetric around m. Thus, in this case, the W smoother is an unbiased estimator of the mean, and $E[Y(i,j)] = E[X(i,j)] = m$.

Using (5), the output variances were computed through numerical integration for i.i.d. Gaussian inputs with mean zero and variance one ($N(0,1)$). The results associated with the W(9,a) smoother are tabulated in Table 1. For comparison, the variances of median filters with different window sizes are also shown. As expected, among W smoothers, the W(9,5) smoother, which is the median filter with window size nine, performs the best, and the output variance of the W smoother increases as parameter a decreases. When $a = 4$ (= 3), the W smoother performs better than the median filter with size 5 (3) but worse than that with size 7 (5).

The impulse noise suppression characteristic can be analyzed statistically from Property 3. We evaluated the *breakdown probablity* [6] from the output distribution in (5) by assuming a binomial input. Here, roughly speaking, the breakdown probability is the probablity of an impulse occuring at the output. This probablity is computed from the output distribution in (5) by assuming a binomial input. Table 2 shows the breakdown probablity of W(9,a) smoothers, and those of median filters. Again it is seen that the W smoother with $a = 4$ (= 3) performs better than the median filter with size 5 (3) but worse than that with size 7 (5).

Table 1 — Output Variance

a	W(9,a) smoother	2L+1	Median
2	0.673	3	0.449
3	0.415	5	0.287
4	0.237	7	0.210
5	0.166	9	0.166

Table 2 — Breakdown Probability

a	W(9,a) smoother		2L+1	Median	
	p=0.0625	p=0.125		p=0.0625	p=0.125
2	0.02521	0.08205	3	0.01123	0.04297
3	0.00531	0.03296	5	0.00222	0.01605
4	0.00067	0.00849	7	0.00046	0.00624
5	0.00010	0.00248	9	0.00010	0.00248

Table 2. Breakdown probabilities when the input is binomial, B(2L+1,p).

B. Edge Preservation

We now consider the effects of W smoothers on noisy step edges. The 2-D input sequence representing a noisy step edge is expressed by

$$X(i,j) = \begin{cases} V(i,j), & j \leq 0 \\ h + V(i,j), & j \geq 1 \end{cases} \quad (6)$$

where h is a constant representing edge height, $V(i,j)$ is i.i.d. noise with distribution $F_1(x)$. Let the distribution function of $h + V(i,j)$ be $F_2(x)$. Then, obviously, $F_2(x) = F_1(x - h)$.

We will examine the filter behavior near the noisy edge by using the expected value of the output and the root mean squared error (rmse). Here the rmse at (i,j) denoted by rmse(i,j), is defined as rmse$(i,j) = \sqrt{E[Y(i,j) - S(i,j)]^2}$ with $Y(i,j)$ as the filtered output, $S(i,j)$ equal to 0 if $j \leq 0$, and equal to h if $j \geq 1$. In order to compute these quantities, we derive the distribution function $F_{\mathbf{Y}_{ij}}(y)$ of the output $Y(i,j)$ of the W smoother which is taken from m_{ij} samples with distribution $F_1(x)$ and $2L+1-m_{ij}$ samples with distribution $F_2(x)$ among $2L + 1$ samples within the window centered at (i,j). Note that the number of samples having $F_1(x)$ in the window, m_{ij}, depends on the location of the window.

Property 4: For the noisy step edge input in (6), the output distribution function $F_{\mathbf{Y}_{ij}}(y)$ of the W(2L+1,a) smoother is given by

$$F_{\mathbf{Y}_{ij}}(y) = \sum_{k=k_1-1}^{2L} \sum_{l=max(0,k-(2L-d))}^{min(k,d)} \binom{d}{l} \binom{2L-d}{k-l} F_1^l(y)(1-F_1(y))^{d-l} F_2^{k-l}(y)(1-F_2(y))^{2L-d-k+l} F_{ij}(y) + \sum_{k=k_2}^{2L} \sum_{l=max(0,k-(2L-d))}^{min(k,d)} \binom{d}{l} \binom{2L-d}{k-l} F_1^l(y)(1-F_1(y))^{d-l} F_2^{k-l}(y)(1-F_2(y))^{2L-d-k+l}(1-F_{ij}(y)), \quad (7)$$

where $k_1 = a, k_2 = 2L + 2 - a$,

$$F_{ij}(y) = \begin{cases} F_1(y), & j \leq 0 \\ F_2(y), & j \geq 1, \end{cases} \quad \text{and} \quad d = \begin{cases} m_{ij} - 1, & j \leq 0 \\ m_{ij}, & j \geq 1 \end{cases}$$

Using (7), along with the assumption that $F_1(x)$ is N(0,1), we computed $E[Y(i,j)]$ and rmse(i,j) along the horizontal filter path (fixed i, variable j) through numerical integration. Fig. 1 (a) and (b) show plots of $E[Y(i,j)]$ and rmse(i,j), respectively, for the W smoothers with the 3x3 square window, when the step edge with $h = 4$ is degraded by a Gaussian $N(0,1)$ noise. It is interesting to see that the W(9,2) smoother has the expected values closest to the ideal step edges, but has the largest rmse values due to its poor noise suppression characteristics. The results show that all of the filters are essentially edge preserving filters.

C. Line Preservation

The detail preservation characteristic of a W smoother can be examined by considering a noisy line image $X(i,j)$ defined by

$$X(i,j) = \begin{cases} V(i,j), & j \neq 0 \\ h + V(i,j), & j = 0 \end{cases} \tag{8}$$

where h is the height of the line, $V(i,j)$ is i.i.d. with distribution $F_1(x)$.

For the noisy line input in (8), the output distribution function $F_{Y_{ij}}(y)$ of the W(2L+1,a) smoother is the same as (7) except that

$$F_{ij}(y) = \begin{cases} F_1(y), & j \neq 0 \\ F_2(y), & j = 0, \end{cases} \quad \text{and} \quad d = \begin{cases} m_{ij} - 1, & j \neq 0 \\ m_{ij}, & j = 0. \end{cases}$$

Again, under the assumption that $F_1(x)$ is N(0,1), we computed $E[Y(i,j)]$ and rmse(i, j) for the noisy line along the horizontal filter path through numerical integration. The results associated with the W smoothers with a 3x3 square window are plotted in Fig. 2 (a) and (b) for $h = 4$. It is seen that W smoothers with $a = 3$, and 2 preserve the line, while the others remove it.

In summary, the W smoother can preserve edges and details while reducing noise. However, there exists a clear tradeoff between detail preservation and noise suppression properties of this smoother. The parameter a should be carefully selected depending on both the characteristics of the input image and its noise. The results presented in this section provide some criteria for determining the parameter of W smoothers. For example, suppose that an image is corrupted by impulses which occur with probability 0.0625, and that we wish to remove at least 99% of the impulses. If a W smoother with a 3x3 square window is used, then its parameter should be greater than or equal to four (Table 2). If the image has lines to be preserved, then the W(9,2) smoother with a 3x3 square window would be satisfactory. (Statistical results for CWM filters with a 5x5 square window are available in [8].)

IV. Experimental Results

The performance of the filters discussed so far is evaluated by applying them to noisy images degraded by additive white and/or impulsive noise and then by comparing their respective results. The original noise-free image is shown in Fig. 3. Three noisy images were generated by adding zero mean i.i.d. Gaussian noise of variance 100, 200, and 400 to the original image, and then were passed through various filters with 5x5 square window. In the following, we first compare the normalized mean square error (NMSE) between the original and filtered images, and then visually compare some of the filtered images.

Fig.4 exhibits the NMSE's associated with W(25,a) smoothers. It is interesting to observe that the NMSE curve depicted as a function of a is convex and has a unique minimum value at a certain value of a. The parameter a which minimizes the NMSE is dependent on the noise variance: it becomes smaller as the noise variance decreases. In general, for a given image to be W smoothed with a certain window, it is most prudent to apply all W smoothers with $1 \leq a \leq L + 1$, and then to choose one yielding the best result. Table 3 summarizes the NMSE's of the W smoothers and median-type filters. In each case, the minimal NMSE of a W smoother is smaller than the NMSE of a median filter. The NMSE's of the multi-stage median filters, with the exception of the one associated with variance 100, are larger than those measure of the W smoothers with minimal NMSE.

Next we compare the performance of the filters in reducing impulsive noise. The noisy image with the noise variance 200 was further corrupted by both positive and negative impulses having values 255 and 0, respectively. By considering two different probabilities of an impulse occurring, $p = 0.02$ and $p = 0.1$, two noisy images were generated, and then passed through the filters. (The probabilities of occurrence of a positive and a negative impulse are the same.) Table 4 shows the resulting NMSE's. The noisy image degraded by both Gaussian noise $\sigma^2 = 200$ and impulses $p = 0.02$ and the filtered images (5x5 square window) are shown in Fig. 5. (The other filtered images are not presented because they lead to a discussion similar to the one stated below.) Comparison of these images clearly indicates that the median filter performs the worst, and that the W smoothed and multi-stage median filtered images appear alike. It is clearly seen that the W smoothers suppress impulses while preserving image details.

The results in this section indicate that the W smoother is an effective detail-preserving filter that can suppress additive white and impulsive noise.

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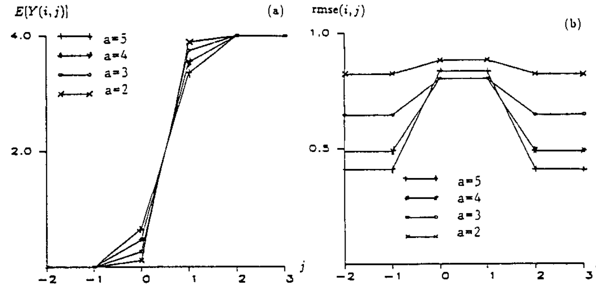

Fig. 1. Results of median and W smoothers, with 3x3 square window, for the noisy step edge $h = 4$: (a) The output expected values, (b) The root mean square errors.

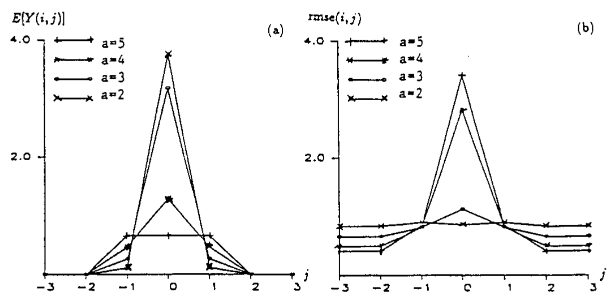

Fig. 2. Results of median and W smoothers, with 3x3 square window, for the noisy line $h = 4$: (a) The output expected values, (b) The root mean square errors.

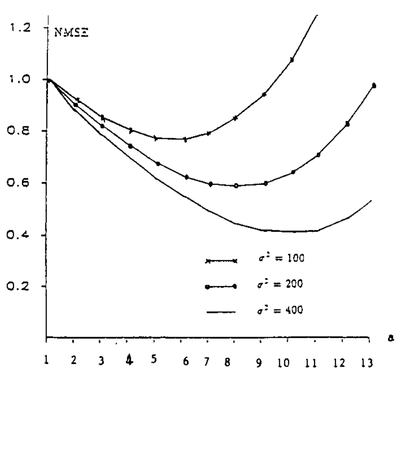

Fig. 4. NMSE's of W smoothers with 5x5 square window.

NMSE (Noise type: Gaussian)				
Filter Type (5x5)		σ^2 =100	σ^2 =200	σ^2 =400
Median		1.86	0.98	0.54
W(25,a) smoother	a=10	1.07	0.64	0.41*
	a=8	0.85	0.59*	0.45
	a=6	0.77*	0.63	0.55
Multi-stage median		0.72	0.62	0.55

Table 3. NMSE's associated with Gaussian noise. (* indicates the minimal NMSE of W smoothers.)

NMSE (Noise Type: Both Gaussian and impulsive)			
Filter Type (5x5)		p=0.02 & σ^2=200	p=0.1 & σ^2 =200
Median		0.51	0.173
W(25,a) smoother	a=10	0.33	0.119*
	a=8	0.31*	0.123
	a=6	0.32	0.148
Multi-stage median		0.34	0.153

Table 4. NMSE's associated with both Gaussian and impulsive noise. (* indicates the minimal NMSE of W smoothers.)

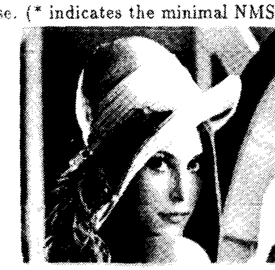

Fig. 3. Original image.

Fig. 5. (a) Noisy image

Fig. 5. (b) median filter

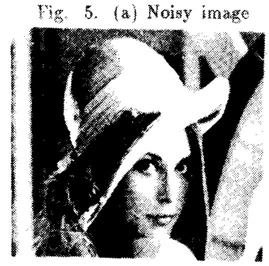

Fig. 5 (c) muti-stage median

Fig. 5. (d) W(25,8) smoother