Output Distribution of Recursive Median Filters

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Abstract: Median filters have been known as more effective smoothers than linear filters for some applications. In particular, the median filter preserves edges in signals, which removes impulsive noise. Understanding of statistical properties of median filters is therefore important and interesting. In this paper, we redefine the output states of recursive median filters. Using statistical threshold decomposition and the redefined states, the output cumulative distribution function of recursive median filters useful for any input distribution is derived.

1. Introduction

Median filter was first introduced by Tukey [1]. One of the main characteristics of the median filter is that it has a low-pass characteristic yet preserves edges. *Recursive median filter* [2] as the recursive version of the median filter has some better properties. The output $y(m)$ of a recursive median filter of window size $2N+1$ is

$$y(m)=\text{med}\{y(m-N),\dots,y(m-1),a(m),\dots,a(m+N)\}, \tag{1}$$

where $\{a(m)\}$ is the discrete-time input sequence.

Assume that the input $a(m)$ is quantized to one of the k values, $0,1,\cdots,k-1$. The thresholded binary signal $t^j(m)$ is defined by

$$t^j(m)=\begin{cases}1, & \text{if } a(m)\geq j\\ 0, & \text{if } a(m)<j\end{cases}, \tag{2}$$

where $1\leq m\leq L$, $1\leq j\leq k-1$, and L is the length of the input sequence. For the thresholded input sequence $\{t^j(m)\}$,

$$x^j(m)=\text{med}\{x^j(m-N),\dots,x^j(m-1),t^j(m),\dots,t^j(m+N)\}. \tag{3}$$

In this paper, we obtain the cumulative distribution function of sequences filtered by recursive median filters, where the output of a recursive median filter is reconstructed by

$$y(m)=\sum_{j=1}^{k-1} x^j(m) \tag{4}$$

2. First Order Output Distributions of Recursive Median Filters

Statistical analysis of nonrecursive median filter is based usually on the well-known theory of order statistics [3]. On the other hand, statistical analysis of recursive median filters is based on the *statistical threshold decomposition* [4].

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Property 1 [4] : If $\{x^j(m)\}$ is the recursive median filter output sequence for a thresholded binary input sequence, then

$$P\{x^j(m)=0\}=P\{y(m)<j\}=F_y(j-1), \tag{5}$$

where $F_y(\cdot)$ is the cumulative distribution function of the recursive median filter output sequence $\{y(m)\}$.

Using Property 1 and the states defined in [5], a sequence of N+1 consecutive samples, the first order marginal probabilities of recursive median filter output was derived [6] when the input sequence was independent and identically distributed (i.i.d.) and first order Markov chains.

Recently it was found [7] that only the immediate previous output $y(m-1)$ and future inputs are necessary to determine the present output $y(m)$. That is,

$$y(m)=\text{med}\{y(m-1),a_{\min}(m),a_{\max}(m)\}, \tag{6}$$

where $a_{\min}(m)=\min\{a(m), a(m+1),\dots,a(m+N)\}$ and $a_{\max}(m)=\max\{a(m), a(m+1),\dots,a(m+N)\}$. Therefore, the recursive median filter output $x^j(m)$ for a thresholded input sequence $\{t^j(m)\}$ is given by

$$x^j(m)=\text{med}\{x^j(m-1),t^j_{\min}(m),t^j_{\max}(m)\}, \tag{7}$$

where $t^j_{\min}(m)=\min\{t^j(m), t^j(m+1),\dots,t^j(m+N)\}$ and $t^j_{\max}(m)=\max\{t^j(m), t^j(m+1),\dots,t^j(m+N)\}$.

Since only one previous output $x^j(m-1)$ and future inputs are necessary to determine the present output $x^j(m)$, it is easy to see that there may exist some state transitions with probability 1 among those defined in [4]. Based on this observation, we redefined the threshold-filtered output states of

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recursive median filters.

Property 2 : The threshold-filtered output transitions of a recursive median filter are specified by the four states $\{x^j(m)=0|x^j(m-1)=0\}$, $\{x^j(m)=1|x^j(m-1)=0\}$, $\{x^j(m)=0|x^j(m-1)=1\}$, and $\{x^j(m)=1|x^j(m-1)=1\}$.

One of the significance of Property 2 is that the states are now defined independent of window size. Using these redefined states, we can simplify the theoretical analysis of recursive median filters.

Since $\{t^j_{\min}(m)=0,\ t^j_{\max}(m)=0\}$ = $\{t^j_{\max}(m)=0\}$, $\{t^j_{\min}(m)=1,\ t^j_{\max}(m)=1\}$ = $\{t^j_{\min}(m)=1\}$, $\{t^j_{\min}(m)=0,\ t^j_{\max}(m)=1\}$ = $\{\{t^j_{\max}(m)=0\} \cup \{t^j_{\min}(m)=1\}\}^C$ and $\{t^j_{\min}(m)=1,\ t^j_{\max}(m)=0\} = \phi$, and $P\{x^j(m)=0|t^j_{\max}(m)=0,\ x^j(m-1)=0\} = 1$, $P\{x^j(m)=0|t^j_{\min}(m)=1,\ x^j(m-1)=0\} = 0$, and $P\{x^j(m)=0|t^j_{\min}(m)=0,\ t^j_{\max}(m)=1,\ x^j(m-1)=0\} = 1$, the transition probability $P\{x^j(m)=0|x^j(m-1)=0\}$ can be written as

$$
\begin{aligned}
&P\{x^j(m)=0|x^j(m-1)=0\}\\
&\quad=P\{x^j(m)=0|t^j_{\max}(m)=0,x^j(m-1)=0\}\\
&\qquad\times P\{t^j_{\max}(m)=0|x^j(m-1)=0\}\\
&\qquad+P\{x^j(m)=0|t^j_{\min}(m)=1,x^j(m-1)=0\}\\
&\qquad\times P\{t^j_{\min}(m)=1|x^j(m-1)=0\}\\
&\qquad+P\{x^j(m)=0|t^j_{\min}(m)=0,t^j_{\max}(m)=1,x^j(m-1)=0\}\\
&\qquad\times P\{t^j_{\min}(m)=0,t^j_{\max}(m)=1|x^j(m-1)=0\}.\\
&\quad=P\{t^j_{\max}(m)=0|x^j(m-1)=0\}\\
&\qquad+P\{t^j_{\min}(m)=0,t^j_{\max}(m)=1|x^j(m-1)=0\}\\
&\quad=P\{t^j_{\min}(m)=0|x^j(m-1)=0\}. \qquad (8)
\end{aligned}
$$

We next express (8) in terms of the joint probabilities of thresholded input sequence $\{t^j(m)\}$. First, from all the possible events, we retain the events $\{t^j(m-1), \cdots, t^j(m+N)\}$ for which $x^j(m-1)=0$ only. Next among those events, we find the subevents for which $t^j_{\min}(m)=0$. Then, $P\{t^j_{\min}(m)=0|x^j(m-1)=0\}$ equals the ratio of the probabilities of the subevents to that of the events for which $x^j(m-1)=0$.

Let the value of $x^j(m-2)$ be *don't care*. Thus $x^j(m-2)$ has all the effects of the previous outputs $x^j(m-3)$, $x^j(m-4)$, $\cdots$, and $x^j(1)$. Since the event for which $x^j(m-1)=0$ can never occur irrespective of the value of $x^j(m-2)$ when $t^j_{\min}(m-1)=1$, the events for which $x^j(m-1)=0$ are the events except the event $t^j_{\min}(m-1)=1$ among all possible events. Since $x^j(m-2)$ is don't care, $P\{x^j(m-1)=0\} = 1 - P\{t^j_{\min}(m-1)=1\}$. Next, since the event for which $t^j_{\min}(m)=0$ never occurs when $t^j(m-1)=0$ and $t^j_{\min}(m)=1$, the events for which $t^j_{\min}(m)=0$ are the events for which $x^j(m-1)=0$ except the event $\{t^j(m-1)=0,\ t^j_{\min}(m)=1\}$. Thus we have $P\{t^j_{\min}(m)=0,\ x^j(m-1)=0\} = 1 - P\{t^j_{\min}(m-1)=1\}$ $P\{t^j(m-1)=0,\ t^j_{\min}(m)=1\}$ when $x^j(m-2)$ is don't care. Therefore, we have

$$
\begin{aligned}
&P\{x^j(m)=0|x^j(m-1)=0\}=P\{t^j_{\min}(m)=0|x^j(m-1)=0\}\\
&\quad=\frac{1-P\{t^j_{\min}(m-1)=1\}-P\{t^j(m-1)=0,t^j_{\min}(m)=1\}}{1-P\{t^j_{\min}(m-1)=1\}}.
\end{aligned} \qquad (9)
$$

Following similar procedures, it can be shown that

$$
\begin{aligned}
&P\{x^j(m)=1|x^j(m-1)=0\}=P\{t^j_{\min}(m)=1|x^j(m-1)=0\}\\
&\quad=\frac{P\{t^j(m-1)=0,t^j_{\min}(m)=1\}}{1-P\{t^j_{\min}(m-1)=1\}},
\end{aligned} \qquad (10)
$$

$$
\begin{aligned}
&P\{x^j(m)=0|x^j(m-1)=1\}=P\{t^j_{\max}(m)=0|x^j(m-1)=1\}\\
&\quad=\frac{P\{t^j(m-1)=1,t^j_{\max}(m)=0\}}{1-P\{t^j_{\max}(m-1)=0\}}
\end{aligned} \qquad (11)
$$

and

$$
\begin{aligned}
&P\{x^j(m)=1|x^j(m-1)=1\} =P\{t^j_{\max}(m)=1|x^j(m-1)=1\}\\
&\quad=\frac{1-P\{t^j_{\max}(m-1)=0\}-P\{t^j(m-1)=1,t^j_{\max}(m)=0\}}{1-P\{t^j_{\max}(m-1)=0\}}.
\end{aligned} \qquad (12)
$$

With the transition probabilities (9)-(12), we can form the Markov transition matrix

$$
\mathbf{P}=\begin{vmatrix} P\{x^j(m)=0|x^j(m-1)=0\} & P\{x^j(m)=0|x^j(m-1)=1\} \\ P\{x^j(m)=1|x^j(m-1)=0\} & P\{x^j(m)=1|x^j(m-1)=1\} \end{vmatrix}
=\begin{vmatrix} 1-\dfrac{P\{t^j(m-1)=0,t^j_{\min}(m)=1\}}{1-P\{t^j_{\min}(m-1)=1\}} & \dfrac{P\{t^j(m-1)=0,t^j_{\min}(m)=1\}}{1-P\{t^j_{\min}(m-1)=1\}} \\ \dfrac{P\{t^j(m-1)=1,t^j_{\max}(m)=0\}}{1-P\{t^j_{\max}(m-1)=0\}} & 1-\dfrac{P\{t^j(m-1)=1,t^j_{\max}(m)=0\}}{1-P\{t^j_{\max}(m-1)=0\}} \end{vmatrix} \qquad (13)
$$

From (13) we obtain $P\{x^j(m)=0\}$ and $P\{x^j(m)=1\}$.

Property 3 : The cumulative distribution function of recursive median filter outputs $F_y(\cdot)$ are given by

$$F_y(j-1)=P\{x^j(m)=0\}$$
$$=\frac{(1-P\{t^j_{\min}(m-1)=1\})\cdot P\{t^j(m-1)=1,t^j_{\max}(m)=0\}}{[(1-P\{t^j_{\max}(m-1)=0\})\cdot P\{t^j(m-1)=0,t^j_{\min}(m)=1\}+(1-P\{t^j_{\min}(m-1)=1\})\cdot P\{t^j(m-1)=1,t^j_{\max}(m)=0\}]} \quad (14)$$

Property 3 allows us to calculate the first order output distribution of recursive median filters when we know the four probabilities $P\{t^j_{\min}(m-1)=1\}$, $P\{t^j_{\max}(m-1)=0\}$, $P\{t^j(m-1)=1,\ t^j_{\max}(m)=0\}$ and $P\{t^j(m-1)=0,\ t^j_{\min}(m)=1\}$.

3. Examples and Simulations

3.1. I.I.D. Input

Let us assume that $P\{t^j(m)=0\} = \alpha_j$, $P\{t^j(m)=1\} = 1-\alpha_j = \beta_j$,and that $\{t^j(m)=0\}$ and $\{t^j(m)=1\}$ are independent. Then we have

$$P\{t^j_{\min}(m-1)=1\} = (\beta_j)^{N+1}, \quad (15)$$

$$P\{t^j_{\max}(m-1)=0\} = (\alpha_j)^{N+1}, \quad (16)$$

$$P\{t^j(m-1)=1,t^j_{\max}(m)=0\} = \beta_j\cdot(\alpha_j)^{N+1} \quad (17)$$

and

$$P\{t^j(m-1)=0,t^j_{\min}(m)=1\} = \alpha_j\cdot(\beta_j)^{N+1}. \quad (18)$$

If we substitute (15)-(18) into (14), we obtain

$$F_y(j-1)=\frac{A(\beta_j,\alpha_j)}{A(\alpha_j,\beta_j)+A(\beta_j,\alpha_j)}, \quad (19)$$

where $A(a,b)=a\cdot(1-a^{N+1})\cdot b^{N+1}$.

Computer simulation was done using a length 100,000 sequence with the uniform distribution $U(0,9)$. The result is shown in Figure 1.

3.2. First Order Markov Chains

Let the thresholded-input transition probabilities be $P\{t^j(m)=1|t^j(m-1)=0\} = p_j$ and $P\{t^j(m)=0|t^j(m-1)=1\} = q_j$. Then we have

$$P\{t^j_{\min}(m-1)=1\} = \frac{p_j}{p_j+q_j}\cdot(1-q_j)^N, \quad (20)$$

$$P\{t^j_{\max}(m-1)=0\} = \frac{q_j}{p_j+q_j}\cdot(1-p_j)^N, \quad (21)$$

$$P\{t^j(m-1)=1,t^j_{\max}(m)=0\} = \frac{p_j\cdot q_j}{p_j+q_j}\cdot(1-p_j)^N \quad (22)$$

and

$$P\{t^j(m-1)=0,t^j_{\min}(m)=1\} = \frac{q_j\cdot p_j}{p_j+q_j}\cdot(1-q_j)^N \quad (23)$$

Substituting (20)-(23) into (14), we obtain

$$F_y(j-1)=\frac{B(p_j,q_j)}{B(p_j,q_j)+B(q_j,p_j)}, \quad (24)$$

where $B(a,b)=(a+b-a(1-b)^N)\cdot(1-a)^N$.

As in the case of the i.i.d input, computer simulation was done using a length 100,000 sequence which was generated by $a(m) = 0.8\times a(m-1) + 4\times z(m)$, where $z(m) \sim U(0,3)$. The result is shown in Figure 2.

3.3. Second Order Markov Chains

Let us assume that $P\{t^j(m)=0\} = \epsilon_j$, $P\{t^j(m)=1\} = \zeta_j$, $P\{t^j(m)=0 \mid t^j(m-1)=0\} = \eta_j$, $P\{t^j(m)=0 \mid t^j(m-1)=1\} = \theta_j$, $P\{t^j(m)=1 \mid t^j(m-1)=0\} = \iota_j$, $P\{t^j(m)=1 \mid t^j(m-1)=1\} = \kappa_j$, $P\{t^j(m)=0 \mid t^j(m-1)=0,\ t^j(m-2)=0\} = \lambda_j$, $P\{t^j(m)=0 \mid t^j(m-1)=0,\ t^j(m-2)=1\} = \mu_j$, $P\{t^j(m)=1 \mid t^j(m-1)=1,\ t^j(m-2)=0\} = \nu_j$, and $P\{t^j(m)=1 \mid t^j(m-1)=1,\ t^j(m-2)=1\} = \xi_j$. Then we have

$$P\{t^j_{\min}(m-1)=1\} = (\xi_j)^{N-1}\cdot\kappa_j\cdot\zeta_j, \quad (25)$$

$$P\{t^j_{\max}(m-1)=0\} = (\lambda_j)^{N-1}\cdot\eta_j\cdot\epsilon_j, \quad (26)$$

$$P\{t^j(m-1)=1,t^j_{\max}(m)=0\} = (\lambda_j)^{N-1}\cdot\mu_j\cdot\theta_j\cdot\zeta_j \quad (27)$$

and

$$P\{t^j(m-1)=0,t^j_{\min}(m)=1\} = (\xi_j)^{N-1}\cdot\nu_j\cdot\iota_j\cdot\epsilon_j. \quad (28)$$

Therefore, if we substitute (25)-(28) into (14), we obtain

$$F_y(j-1)=\frac{C(\xi_j,\kappa_j,\zeta_j,\lambda_j,\mu_j,\theta_j)}{C(\lambda_j,\eta_j,\epsilon_j,\xi_j,\nu_j,\iota_j)+C(\xi_j,\kappa_j,\zeta_j,\lambda_j,\mu_j,\theta_j)} \tag{29}$$

where $C(a,b,c,d,e,f)=\{1-a^{N-1}\cdot b\cdot c\}\cdot d^{N-1}\cdot e\cdot f\cdot c$.

Computer simulation was done using a length 100,000 sequence which was generated by $a(m) = 0.2\times a(m-2) + 0.1\times a(m-1) + 8.8\times z(m)$, where $z(m) \sim U(0,24)$. The result is shown in Figure 3.

From Figures 1-3, it is easy to see that the cdfs calculated by (19), (24) and (29) are very close to those obtained from simulation.

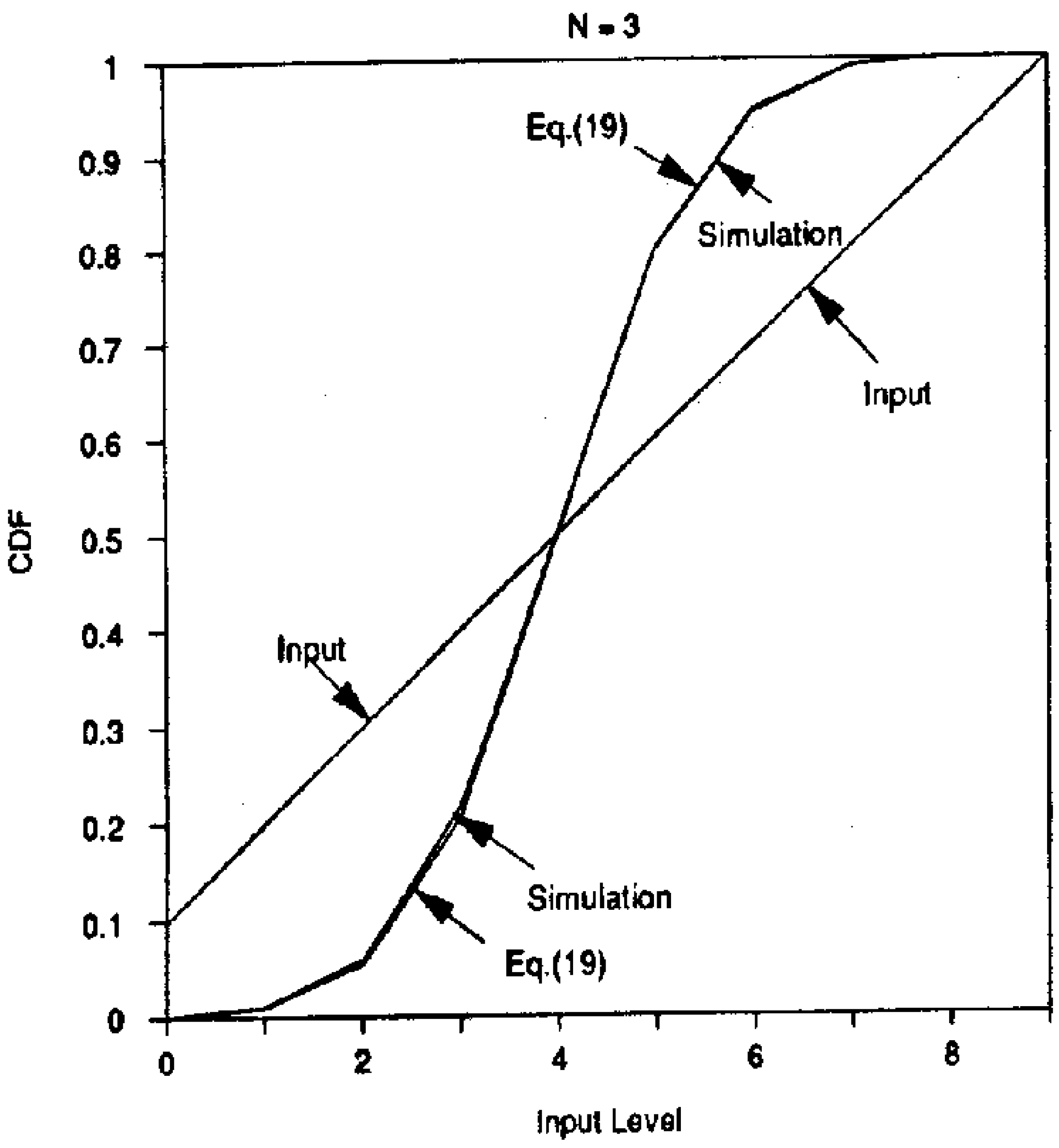

Figure 1. I.I.D. Input

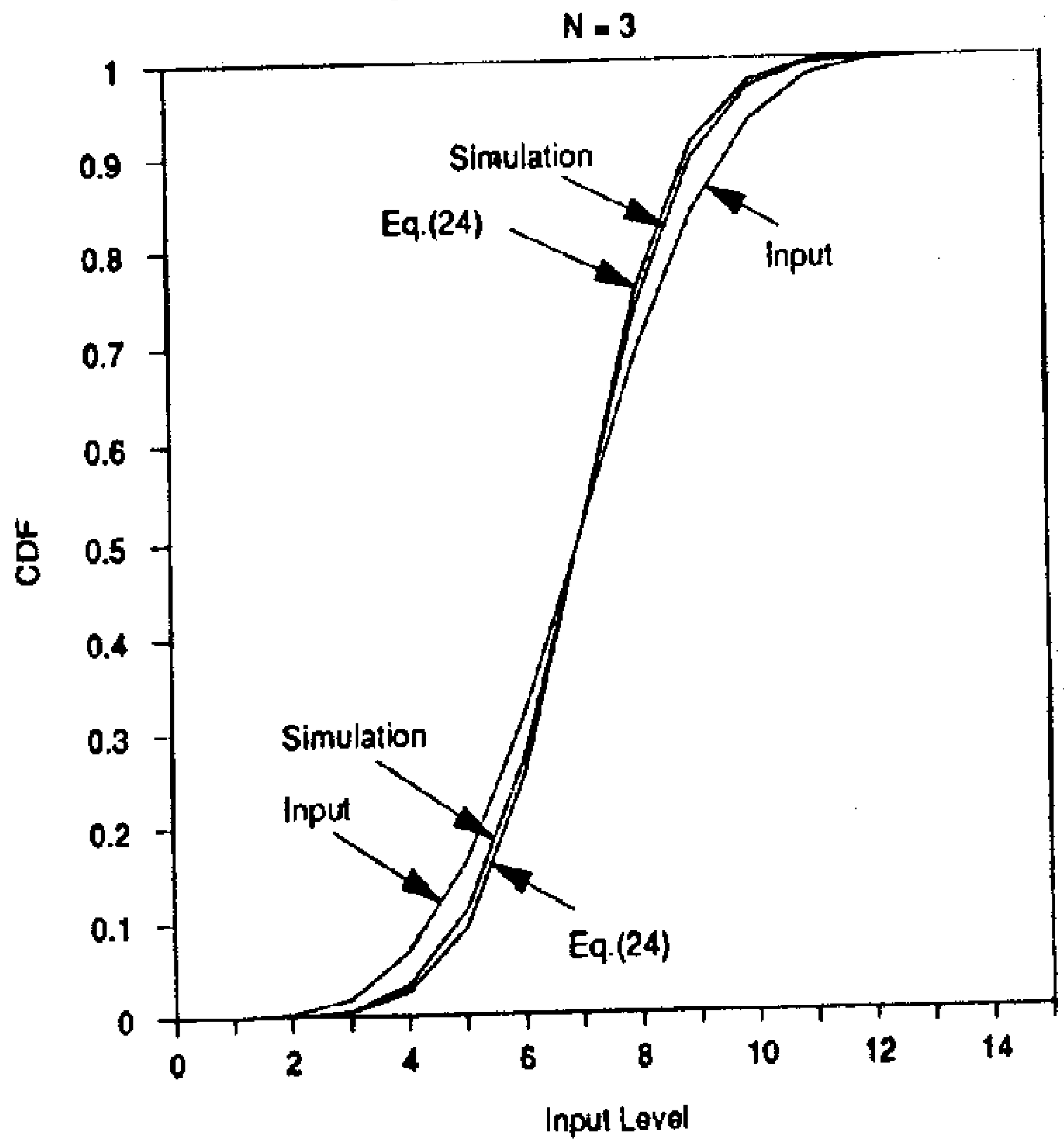

Figure 2. First Order Markov Chains

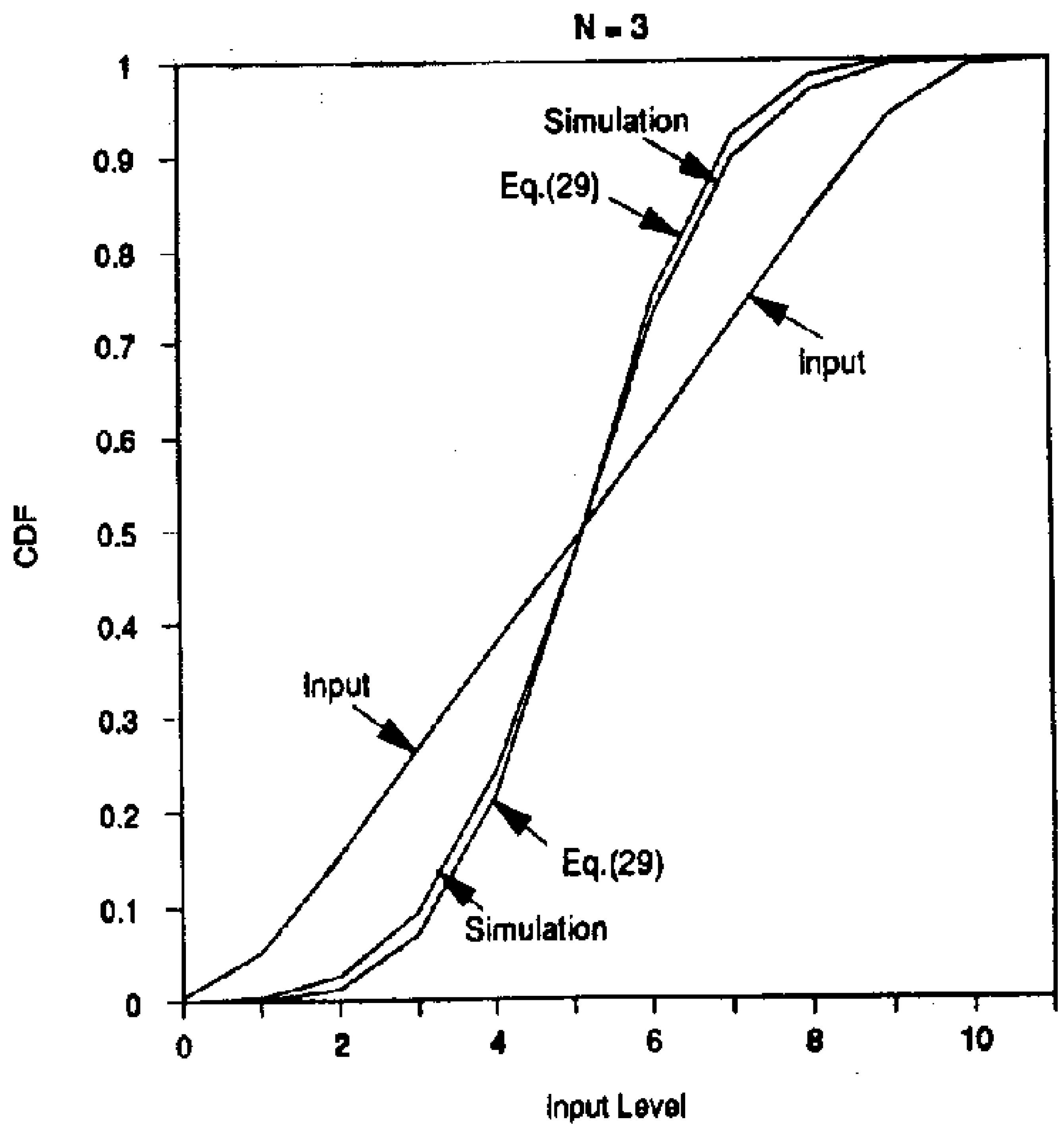

Figure 3. Second Order Markov Chains

4. Conclusion

We have redefined the output states of recursive median filters. The redefined states are independent of window size. Using statistical threshold decomposition and the redefined states, we have derived the cumulative distribution function of recursively median filtered sequences useful for any input distribution. The usefulness of the result was illustrated through computer simulation.

5. References

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