SELECTION FILTERS AND COMMUTATIVITY

WITH MEMORYLESS NONLINEARITIES

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Abstract

A class of nonrecursive filters that commute with every increasing zero-memory nonlinear (ZNL) transformation is characterized. Specifically, it is shown that a nonrecursive filter commutes with every increasing ZNL transformation if and only if it is a rank-based selection (RBS) filter that replaces each input value with one of its neighboring input data which is selected depending on relative amplitudes of the data. We also show that RBS filters commuting with every nondecreasing ZNL transformation are stack filters that can be represented as finite maximum-minimum operations.

I. Introduction

It has been observed that median filters and some of their extension commute with every monotone zero-memory nonlinear (ZNL)[1] transformation [1,2]. As an example, for 1-D median filters, if $\{X(m)\}$ and $\{Y(m)\}$ denote the input and output sequences, respectively, and $f(\cdot)$ denotes a monotone ZNL transformation, then the commutativity implies that

$$f[Y(m)] = \text{median}\{f[X(m-M)], ..., f[X(m)], ..., f[X(m+M)]\} \quad (1)$$

where $Y(m) = \text{median}\ \{X(m-M), ..., X(m), ..., X(m+M)\}$ and $2M+1$ is the window size. The commutative property of a median filter is due to the fact that it selects one of the sample data within each window depending on the relative amplitudes (or the amplitude rank) of the data, and that the amplitude rank is invariant to monotone ZNL transformations. For example, if $x_1 \le x_2 \le x_3$, then $f(x_1) \le f(x_2) \le f(x_3)$ for every monotone ZNL $f(\cdot)$. Maragos and Schafer, in their excellent work on morphological filters [2], showed that some median-type filters commute with thresholding. Here the commutativity with thresholding is essentially the commutativity with unit-step functions. Due to the commutativity, any analytical results in [2] obtained for median-type filters for binary signals were also valid for multi-level signals. In addition, the commutativity played a major role in their proof that stack filters in [3] are morphological filters that can be expressed as finite maximum-minimum operations. Fitch et. al. [4] proposed an interesting median filtering algorithm by showing that median filtering of multi-level signals reduces to a sum of median filters for binary signals. As noted by Maragos and Schafer, this median filtering approach is in fact a consequence of the commutativity with unit-step functions.

Many median-type nonlinear filters select one of the samples inside each window that moves over an input sequence; its output sequence consists of the selected samples. For such filters we propose the name *selection* filters. If the output of a selection filter is selected depending only on the amplitude ranks of the inputs within each window then we call it the *rank-based selection* (RBS) filters. Obviously, median filters and stack filters[2] - which can be represented as finite maximum-minimum operations - are RBS filters. In this paper, we analyze selection and RBS filters in terms of the commutativity with monotone ZNL transformations. In Section II we define rigorously selection and RBS filters and show that the class of stack filters is a subclass of RBS filters. In Section III it is shown that the commutativity with arbitrary increasing ZNL transformations is a defining characteristic of RBS filters. In Section IV the relation between RBS and stack filters is examined further. Specifically, it is shown that the commutativity with arbitrary increasing *and* nondecreasing ZNL transformations is a defining characteristic of stack filters.

Throughout this paper we assume, for simplicity, that the input signals are 1-D. However, the obtained results are also valid for multi-D signals. This is so because in RBS filtering multi-D data after isolation through windowing are treated just like 1-D data - they are ordered and a sample is selected depending on the amplitude rank.

II. Selection and RBS Filters: Definitions and Notations

Consider a nonrecursive discrete filter represented as

$$Y(m) = F\{X_1(m), X_2(m), ..., X_N(m)\} \quad (2)$$

where $Y(m)$ is the output at time m, $X_i(m)$ is the i-th input data from the left of the window at m, and F denotes a filtering operation. It is assumed that input data are real-valued. If the output $Y(m)$ is one of the inputs $X_1(m), ..., X_N(m)$ for all m, then such a filter is called the selection filter. As mentioned in the previous section, a selection filter is an RBS filter provided the selection is done based only on the amplitude rank of the input data. Mathematically, we define an RBS filter using a look-up table which lists all possible ranks of the inputs $X_1, ..., X_N$, and assigns an output - which is one of the inputs - to each rank. Here the inputs are ranked using both strict inequalities and equalities. For example, an RBS filter of span 2 can be specified by assigning outputs to ranks "$X_1 < X_2$", "$X_1 = X_2$" and "$X_1 > X_2$". We call a look-up table that defines an RBS filter a *rank-output* table. In addition, input ranks associated with strict inequalities and equalities are called *strict* ranks. Defining an RBS filter in this manner is rather naive in practice but, as will be seen later, enables us to analyze these filters and introduce RBS filters which are not stack filters. Some examples of selection and RBS filters are presented below.

Example 1 (The median filter) : The median filter of span 3 is given by

$$Y = \text{median}\ \{X_1, X_2, X_3\}^3.$$

Its output is determined if the input rank is given as shown in Table 1(a). Thus the median filter is an RBS filter.

Example 2 (The stack filter) : Consider a stack filter of span 3 defined as

$$Y = \max\ \{\min[X_1, X_2], \min\ [X_1, X_3]\}.$$

Table 1(b) tabulates its output for each input rank. Again, this is an RBS filter.

It is straightforward to see that any stack filter can be represented by a rank-output table, and thus they are RBS filters. There are selection filters which are not RBS. For example, the

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[1] What we call ZNL transformations include zero-memory linear transformations as well.

[2] Throughout this paper stack filters refers to the morphological function and set processing filters that commute with thresholding, which are defined in [2]. This definition of stack filters is a slight extension of its original definition in [3].

filter proposed in [5] adaptively selects one of the data inside each window and cannot be represented by a rank-output table. Thus it is a selection filter but not an RBS filter.

The number of strict ranks which are required to define an RBS filter of span N is $N! + \sum_{i=1}^{N-1} (N-i)! \binom{N}{i+1}$, which will be denoted by N_R. A rank-output table divides Euclidean N-space, with N-D vectors $\mathbf{X} = (X_1, X_2, ..., X_N)$ into N_R subspaces, say $\mathbf{R}_i$, $1 \le i \le N_R$, and assigns an output to each subspace. The subspaces are referred to as *decision regions* and a decision region is called a *boundary* if its specification includes at least one equality. For example, in Table 1 the region specified by "$X_1 < X_2 = X_3$" is a boundary ; furthermore, it is the boundary between the regions "$X_1 < X_2 < X_3$" and "$X_1 < X_3 < X_2$".

Although specifying an RBS filter using strict ranks (or using decision regions which discriminates between boundaries and non-boundaries) is most general, such specification is often redundant: rank-output tables of many RBS filters can be reduced by using greater than or equal to relations ($\le$). For example, the rank-output tables of the median and stack filters in Table 1(a) and 1(b) can be reduced as in Table 2. This is true, because in these filterig if we let R_k be the boundary - with an output X_{k_0} - between decision regions R_i and R_j with outputs X_{i_0} and X_{j_0}, respectively, $1 \le i_0, j_0, k_0 \le N$, then the output at the boundary is either X_{i_0} or X_{j_0} which are equivalent at the boundary (i.e., for all vectors in R_k, $X_{i_0} = X_{j_0} = X_{k_0}$). This observation suggests to consider a class of RBS filters that can be specified using *soft* ranks which are based on greater than or equal to relations ($\le$). We call such an RBS filter an *output-map-continuous* (o.m.c.) RBS filter, since its output varies continuously over the decision regions (its output at any boundary is equivalent to those of neighboring decision regions).

It is obvious that not every RBS filter is o.m.c.; an example is given in Table 1(c). We can show without much difficulty that median filters are o.m.c. Then a natural question is the following: is every stack filter o.m.c. ? This will be answered in Section IV.

ORDERINGS	OUTPUTS	
	(a) Median Filter	(b) Stack Filter
$x_1 \le x_2 \le x_3$	x_2	x_1
$x_2 \le x_1 \le x_3$	x_1	x_1
$x_2 \le x_3 \le x_1$	x_3	x_3
$x_3 \le x_1 \le x_2$	x_1	x_1
$x_3 \le x_2 \le x_1$	x_2	x_2
$x_1 \le x_3 \le x_2$	x_3	x_1

Table 2. Simplified Version of the look-up tables in Table 1 (a) and (b).

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The time index m is dropped from $Y(m)$ and $X_i(m)$ to simplify notation.

ORDERINGS	OUTPUTS		
	(a) Median Filter	* (b)Stack Filter	(c)An RBS Filter
$x_1<x_2<x_3$	x_2	x_1	x_1
$x_1<x_3<x_2$	x_3	x_1	x_1
$x_2<x_1<x_3$	x_1	x_1	x_3
$x_2<x_3<x_1$	x_3	x_3	x_3
$x_3<x_2<x_1$	x_2	x_2	x_2
$x_3<x_1<x_2$	x_1	x_1	x_2
$x_1<x_2=x_3$	x_2	x_1	x_1
$x_2<x_1=x_3$	x_1	x_1	x_3
$x_3<x_2=x_1$	x_2	x_1	x_1
$x_3=x_2<x_1$	x_2	x_2	x_1
$x_1=x_3<x_2$	x_3	x_1	x_2
$x_1=x_2<x_3$	x_2	x_1	x_1
$x_1=x_2=x_3$	x_2	x_1	x_1

Table 1. Look-up tables representing RBS filters of span 3.
$^*Y = \max\{\min[x_1,x_2], \min[x_1,x_3]\}$

III. Selection and RBS filters : Commutativity with ZNL Transformations

Given a filtering operation, it is sometimes possible to find an increasing ZNL transformation that commutes with the filtering. For example: linear filters commute with a linear transformation, say $f(x) = ax$; if $Y(m)$ in (2) is given by the product of inputs, $X_1 X_2 \cdots X_N$, then the operation commutes with $f(x) = x^c$. However, most filters do not commute with arbitrary increasing ZNL transformation. In this section, it is shown that only RBS filters commute with every increasing ZNL transformation. We start with the following lemma.

Lemma 1 : Given an increasing function $f(x)$, $-\infty \le x \le \infty$, we can find an increasing function $g(x)$ in the neighborhood of $f(x)$ such that $g(x_i) = f(x_i)$, $1 \le i \le N$, and $g(y) = f(y)$ where x_i and y are some real numbers and N is a finite, positive integer.

This lemma can be proved easily; its proof is omitted. Using this lemma, a relation between the commutativity and selection filters is derived.

Theorem 1 : If a nonrecursive discrete filter commutes with every increasing ZNL transformation then it is a selection filter.

Proof : Suppose that the filter is not a selection filter. Then there is at least one output such that $Y = F(X_1, ..., X_N) \neq X_i$, for all i, $1 \le i \le N$ [see Eq. (2)]. Consider an increasing ZNL $f(x)$. Then the commutativity implies that $f(Y) = F\{f(X_1), ..., f(X_N)\}$. For the output $Y \neq X_i$, $1 \le i \le N$, we can find an increasing ZNL $g(x)$ in the neighborhood of $f(x)$ such that $g(X_i) = f(X_i)$ for every i but $g(Y) \neq f(Y)$ (Lemma 1). Now $g(Y) = F\{g(X_1), ..., g(X_N)\} = F\{f(X_1), ..., f(X_N)\} = f(Y)$, which is a contradiction. The contradiction does not occur if $Y = X_i$ for some i. Thus the filter must be a selection filter.

The theorem stated below indicates that only RBS filters have the commutative property. To prove the theorem we need the following lemma.

Lemma 2 : Given two input vectors $\mathbf{X}$ and $\hat{\mathbf{X}}$ in the same decision region, we can always find an *increasing* function f(x) which maps $\mathbf{X}$ to $\hat{\mathbf{X}}$ such that $\hat{X}_i = f(X_i)$ for each i, $1 \le i \le N$.

Proof : An increasing function f(x) mapping $\mathbf{X}$ to $\hat{\mathbf{X}}$ can be found as follows. Suppose, without loss of generality, that $X_1 < X_2 < \cdots < X_N$. Then $\hat{X}_1 < \hat{X}_2 \cdots < \hat{X}_N$. Let

$$f(x) = \begin{cases} x + b_1 & \text{for} \quad x \le X_1 \\ a_n x + b_n & \text{for}\ X_{n-1} \le x \le X_n,\ 2 \le n \le N \\ x + b_{N+1} & \text{for} \quad x \ge X_N \end{cases}$$

where $a_n = (\hat{X}_n - \hat{X}_{n-1})/(X_n - X_{n-1})$ and b_n are constants for which $f(X_n) = \hat{X}_n$. Then f(x) is increasing because $a_n > 0$ for all n. This completes the proof.

Theorem 2 : If a selection filter commutes with every increasing ZNL transformation, then it is an RBS filter.

Proof : Suppose that the filter is not RBS. Consider the decision regions based on ranks. Since the filter is not RBS, then there is at least one decision region, say R_i, which is associated with several outputs. Assume, without loss of generality, that R_i assigns two outputs X_n and X_m, $1 \le n$, $m \le N$, $n \neq m$ such that if the input vector is in R_{i_1} then $Y = Y_n$ and in R_{i_2} then $Y = Y_m$ where $R_{i_1} \cup R_{i_2} = R_i$. Consider two input vectors $\mathbf{X} \in R_{i_1}$ and $\hat{\mathbf{X}} \in R_{i_2}$. Since $\mathbf{X}$ and $\hat{\mathbf{X}}$ lie in the same decision region, then we can find an *increasing* function f(x) such that $\hat{X}_j = f(X_j)$, $1 \le j \le N$ (Lemma 2). Now we get $F\{f(X_1), ..., f(X_N)\} = F\{\hat{X}_1, ..., \hat{X}_N\} = \hat{X}_n$. But the commutativity implies that $F\{f(X_1), ..., f(X_N)\} = f(X_m) = \hat{X}_m$, which is a contradiction. Thus the filter must be an RBS filter.

Following from Theorems 1 and 2, we draw the following conclusion.

Theorem 3 : A nonrecursive discrete filter is an RBS filter if and only if it commutes with every increasing ZNL transformation.

Proof : Sufficiency follows from Theorems 1 and 2. Necessity is obvious because the rank of the data within each window is invariant to every increasing ZNL transformation.

This theorem shows that the commutativity is a defining characteristic of RBS filters. The commutativity may play as fundamental a role in analyzing nonlinear RBS filters as the superposition has played in analyzing linear filters.

For the class of stack filters, which is a subclass of RBS filters, the following result holds.

Lemma 3 : Stack filters commute with every nondecreasing ZNL transformation as well as increasing ones.

Proof : This is a direct consequence of the following facts: any stack filter can be represented as finite maximum-minimum operations [2], and maximum and minimum operations commute with every nondecreasing ZNL transformation [1].

In the next section, we shall show that the commutativity with every nondecreasing ZNL transformation is a defining characteristic of stack filters.

IV. Output-Map-Continuous RBS Filters

As described in Section II, the median and stack filters in Table 1 are o.m.c. In addition, Lemma 3 shows that they commute with every nondecreasing ZNL transformation. A question that may arise is the following: is there a relation between the output-map continuity and the commutativity with nondecreasing functions ? The answer is affirmative, as shown in the theorem stated below. Before describing the theorem two lemmas are presented. The first one will give a sufficient condition for an RBS filter being o.m.c., and the second one is a variation of Lemma 2.

Lemma 4 : In RBS filtering, a nonboundary decision region, say R_i with an output X_{i_0}, and its boundary, say R_k with an output X_{k_0}, can be combined to yield a single region - based on soft ranks - which is associated with an output X_{i_0} if for every $\mathbf{X} \in R_i$ and $\hat{\mathbf{X}} \in R_k$ we can find a function f(x), which maps $\mathbf{X}$ into $\hat{\mathbf{X}}$, such that $f(X_{i_0}) = \hat{X}_{k_0}$, where $1 \le i_0, k_0 \le N$.

Proof : Since $f(X_{i_0}) = \hat{X}_{i_0}$, then $\hat{X}_{i_0} = \hat{X}_{k_0}$. This holds for any vector $\hat{\mathbf{X}} \in R_k$, and thus R_i and R_k can be combined.

Obviously, an RBS filter is o.m.c. if this lemma holds for every decision region.

Lemma 5 : Given input vectors $\mathbf{X} \in R_i$ and $\hat{\mathbf{X}} \in R_k$, we can always find a nondecreasing - but *not* strictly increasing - function f(x) that maps $\mathbf{X}$ to $\hat{\mathbf{X}}$, where R_i is a nonboundary decision region, and R_k is its boundary.

This can be proved through slight modification of the proof of Lemma 2; the proof is omitted.

Theorem 4 : An RBS filter is o.m.c. if and only if it commutes with every nondecreasing transformation.

Proof : Sufficiency. Let R_k be the boundary - with an output X_{k_0} - between decision regions R_i and R_j with outputs X_{i_0} and X_{j_0}, respectively, where i_0, j_0 and k_0 are arbitrary integers between 1 and N. Consider two input vectors $\mathbf{X} \in R_i$ and $\hat{\mathbf{X}} \in R_k$. Let f(x) be a nondecreasing function that maps $\mathbf{X}$ to $\hat{\mathbf{X}}$ (Lemma 5). Now $f(X_{i_0}) = F\{f(X_1), ..., F(X_N)\} = F\{\hat{X}_1, ..., \hat{X}_N\} = \hat{X}_{k_0}$, where the first equality follows from the commutativity. Now Lemma 4 indicates that the filter is o.m.c.

Necessity. Suppose that an RBS filter is o.m.c. but it does not commute with every nondecreasing ZNL transformation. Then $F\{f(X_1), ..., f(X_N)\} \neq f(X_{i_0})$ but $F\{f(X_1), ..., f(X_N)\} = F\{\hat{X}_1, ..., \hat{X}_N\} = \hat{X}_{k_0}$. Thus $f(X_{i_0}) = \hat{X}_{i_0} \neq \hat{X}_{k_0}$ which is a contradiction. This completes the proof.

Now we shall examine the relation between o.m.c. RBS filters and stack filters.

Lemma 6 : An o.m.c. RBS filter can be represented by a positive Boolean function when its input is binary.

Proof : For binary inputs, we can construct a truth table corresponding to the rank-output table of a given o.m.c. RBS filter. Obviously, the truth table can be represented by a sum of products Boolean expression where each product term corresponds to a set of binary input vectors producing one as their outputs. Consider a binary vector $(X_1, ..., X_N)$ whose output is one. Let $X_i = 1$ if $i \in A$ and $X_j = 0$ if $j \in B$ where $A \cup B = \{1, 2, ..., N\}$ and $A \cap B = \phi$. Then the o.m.c. condition implies that in the rank-output table every ordering with $X_j \le X_i$ for all $i \in B$, produces one of $\{X_i | i \in A\}$ as its output. This, in turn, indicates that in the truth table the output of every binary vector with $X_i = 1$ for all $i \in A$ and $X_j = 1$ or 0 for all $j \in B$ is equal to one. Now we can see that $\prod_{i \in A} X_i$ is a product term of the Boolean expression. Notice that the product term does not contain any complements of the input variables. In this manner, it can be shown that the Boolean function is positive.

Theorem 5 : An RBS filter is a stack filter if and only if it is o.m.c.

Proof : Sufficiency. This is a direct consequence of Lemma 6 and the fact that an o.m.c. RBS filter commutes with thresholding (Theorem 4).
Necessity. The assumption that the filter is a stack filter implies that it commutes with every nondecreasing ZNL transformation (Lemma 3). Consider two vectors $\mathbf{X} \in R_i$ and $\hat{\mathbf{X}} \in R_k$. We can find a nondecreasing function f(x) such that $f(X_n) = \hat{X}_n$ for every n, $1 \leq n \leq N$ (Lemma 5). Then $\hat{X}_{i_0} = f(X_{i_0}) = F\{f(X_1), ..., f(X_N)\} = F\{\hat{X}_1, ..., \hat{X}_N\} = \hat{X}_{k_0}$. Since this relation holds for every vector in any boundary R_k, the filter is o.m.c.

Following from Theorems 3, 4 and 5 we get the conclusion described below.

Corollary : A nonrecursive discrete filter is a stack filter if and only if it commutes with every nondecreasing transformation .

V. Conclusion

It is shown that the commutativity with every nondecreasing ZNL transformation is a defining characteristic of stack filters. In [2], [4] the commutativity with thresholding (or unit-steps) has been successfully applied to analyze and to implement stack filters. On the other hand, the commutativity with nondecreasing functions other than unit-steps has not been used. Application of the commutative property requires further research.

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