SOME STATISTICAL PROPERTIES OF WEIGHTED MEDIAN FILTERS

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ABSTRACT

In this paper, based on the fact that the output of a weighted median (WM) filter is always one of the samples in the input window, *rank* and *sample selection probabilities* are defined. The former is the probability that a certain ranked sample will appear as the output and latter is the probability that the output equals one of the time-indexed samples. Using the rank selection probabilities, it is shown here that the output distribution of the WM filter of size N with independent identically distributed (i.i.d.) inputs is a weighted sum of the distributions of the $i^{th}, i = 1, 2, \ldots, N$ order statistics. The weights are given by the rank selection probabilities. The sample selection probabilities are the coefficients of the finite impulse response (FIR) filter whose output, of all linear filters, is closest to that of the WM filter. Several statistical properties of WM fiters using selection probabilities are then derived. A method to compute the selection probabilities from the weights of the WM filter is also given.

1. INTRODUCTION

The weighted median (WM) filter can be analyzed under two broad categories viz. deterministic and statistical. An interesting body of literature has accumulated regarding the deterministic properties of WM filters, most notably those by Brownrig,[1], 2 Yli-Harja et. al,[3] and Prasad and Lee.[4] On the other hand, statistical analysis of WM filters, except for the median filter, has gone relatively unnoticed. In their work[3] Yli-Harja et. al analyzed some statistical properties by using its representation as a stack filter.[5] Ko and Lee[6] derived the output distribution of center weighted median filter for i.i.d. inputs without using the stack filter representation. However, these results either lack intuitive appeal or are not general enough.

In this paper we present a method for statistical analysis of WM filters which departs significantly from previous work. First, it is noted that the output of a WM filter is always one of the samples in the input window. Based on this observation *selection probabilities* — *rank* and *sample* selection probabilities — are defined. These are the probabilities that a sample of a certain rank or time-index appears as the output. The rank selection probabilities can be related to the output distribution and the sample selection probabilities give information about the approximate 'frequency response' of the WM filter with i.i.d. inputs.

The rest of this paper is organised as follows. In Section 2 rank and sample selection probabilities are introduced and a number of statistical properties of WM filters are derived. In Section 3 a method for computing the selection probabilities of a WM filter from its weights is given. The concluding remarks in Section 4 sum up the results.

2. SELECTION PROBABILITIES

The output $Y(m)$ of the WM filter of span N is given by

$$Y(m) = \text{median}\{\overbrace{X_1(m), \ldots, X_1(m)}^{w_1 \text{ times}}, \overbrace{X_2(m), \ldots, X_2(m)}^{w_2 \text{ times}}, \ldots \overbrace{X_N(m), \ldots, X_N(m)}^{w_n \text{ times}}\} \quad (1)$$

where $w_j, j = 1, 2, \ldots, N$ are positive integers which sum up to an odd value and $X_j(m)$ is the real-valued sample at the j^{th} position from the left of the window centered at time m. From (1) it is clear that the output of the WM filter is always one of the samples in the input window. Further, since the output is the median of the samples $X_j, j = 1, 2, \ldots, N$ each replicated w_j times, it can be determined if the rank of the samples is known. It is quite natural, therefore, to ask what the probability will be of, say $X_{(i)}, 1 \leq i \leq N$ the i^{th} smallest sample in the window being the output, and how likely will it be for, say $X_j, 1 \leq j \leq N$ the j^{th} sample in the window to appear as the output? Taking our cue from this we will define rank and sample selection probabilities and indicate their importance in describing the statistical behaviour of WM filters.

The remainder of this section is organised as follows. Pertinent notations are introduced in section 2.1. In section 2.2 we consider the relation between rank selection probabilities and WM filters and derive some statistical properties of WM filters based on rank selection probabilities. In section 2.3 sample selection probabilities are considered.

2.1. Notations and Definitions

Consider the definition of WM filters in (1). Let $\mathbf{w} = (w_1, w_2, \ldots, w_N)$ denote the weight vector with positive integer entries $w_j, j = 1, 2, \ldots, N$ such that $\sum_{j=1}^{N} w_j = 2M + 1$, M a positive integer, and let $\mathbf{X} = (X_1, X_2, \ldots, X_N)$ denote the real-valued samples in the input window. The i^{th} smallest sample in $\mathbf{X}$ is denoted by $X_{(i)}, i = 1, 2, \ldots, N$, such that $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(N)}$ and the weight associated with $X_{(i)}$ is $w_{(i)}$. When the input is i.i.d. a sample may take any rank and there are $N!$ ways that an input may be ordered. Let $z_k, k = 1, 2, \ldots, N!$ represent any such ordering. For example, with $N = 2$ '$X_1 \leq X_2$' and '$X_2 \leq X_1$' may be denoted by z_1 and z_2 respectively. Cases with $X_1 = X_2$ may be denoted as either z_1 or z_2. The choice does not affect the value of the output. We shall assume without loss of generality that the ordering associated with each z_k is fixed and known. Any ordering z_k, for which the output $Y = X_{(i)} = X_j, 1 \leq i, j \leq N$, the sets $\mathcal{G}_{ij}^k$ and $\mathcal{H}_{ij}^k$ are defined as follows: $\mathcal{G}_{ij}^k = \{X_{(n)} \mid n = 1, 2, \ldots, i-1, X_{(i)} = X_j$, for the ordering $z_k\}$ and $\mathcal{H}_{ij}^k = \{X_{(n)} \mid n = i+1, i+2, \ldots, N, X_{(i)} = X_j$, for the ordering $z_k\}$. To avoid excessive notations the superscript k will be omitted from $\mathcal{H}_{ij}^k$ and $\mathcal{G}_{ij}^k$ in most cases. Unless otherwise mentioned, the inputs will be assumed to be i.i.d. with probability density $f_X(.)$ and distribution $F_X(.)$. The density and distribution functions of the i^{th} ranked sample are denoted by $f_i(.)$ and $F_i(.)$ respectively, and are given as follows

$$f_i(y) = \frac{N!}{(i-1)!\,(N-i)!} F_X^{i-1}(y)(1 - F_X(y))^{N-i} f_X(y) \tag{2a}$$

$$F_i(y) = \sum_{r=i}^{N} \binom{N}{r} F_X^r(y)(1 - F_X(y))^{N-r} \tag{2b}$$

Selection probabilities are now defined.

<u>Definition 1</u> *Rank Selection Probability (RSP):* The i^{th} rank selection probability is denoted by $P(Y = X_{(i)}), 1 \leq i \leq N$ and is the probability that the output $Y = X_{(i)}$.

<u>Definition 2</u> *Sample Selection Probability (SSP):* The j^{th} sample selection probability is denoted by $P(Y = X_j), 1 \leq j \leq N$ and is the probability that the output $Y = X_j$.

Where necessary, the RSP's $P(Y = X_{(i)}), i = 1, 2, \ldots, N$ will be denoted by the row vector $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ where $r_i = P(Y = X_{(i)}), i = 1, 2, \ldots, N$ and the SSP's $P(Y = X_j), j = 1, 2, \ldots, N$ will be denoted by the row-vector $\mathbf{s} = (s_1, s_2, \ldots, s_N)$ where $s_j = P(Y = X_j), j = 1, 2, \ldots, N$. In the text we will often refer to the number of distinct ways to form the sets $\mathcal{G}_{ij}$ and $\mathcal{H}_{ij}$ such that the output $Y = X_{(i)} = X_j$ and denote this number by C_{ij}. The samples within the sets $\mathcal{G}_{ij}$ and $\mathcal{H}_{ij}$ can be permuted in $(i-1)!$ and $(N-i)!$ ways respectively. Thus the number of orderings for which $Y = X_{(i)} = X_j$, denoted P_{ij}, is given by

$$P_{ij} = (i-1)!\,(N-i)!\,C_{ij} \tag{3}$$

Since each input ordering is equally likely, the *joint selection probability*, denoted $P(Y = X_{(i)}, Y = X_j)$, $1 \leq i, j \leq N$ and defined as the probability that the output equals the i^{th} ranked sample and the j^{th} sample simultaneously, is given by

$$P(Y = X_{(i)}, Y = X_j) = \frac{P_{ij}}{N!} \tag{4}$$

2.2. Rank Selection Probabilities and the WM filter

If the rank of the samples within the window of a WM filter with weights $\mathbf{w}$ and size N is known then the output sample and its rank can be determined. Thus for any given input ordering $z_k, 1 \leq k \leq N!$, the rank of the sample which appears as the output can be determined uniquely. As an example consider the input ordering $z_k : X_5 \leq X_2 \leq X_3 \leq X_1 \leq X_4$ to a WM filter with $N = 5$ and $\mathbf{w} = (1, 2, 3, 2, 1)$. Here $X_{(1)} = X_5$, $X_{(2)} = X_2$, $X_{(3)} = X_3$, $X_{(4)} = X_1$, and $X_{(5)} = X_4$. It can be easily verified that the output $Y = X_{(3)} = X_3$. The sets $\mathcal{G}_{33}$ and $\mathcal{H}_{33}$ for this ordering are $\{X_2, X_5\}$ and $\{X_1, X_4\}$ respectively. It is interesting to note that if the ranks of elements within $\mathcal{G}_{33}$ and/or $\mathcal{H}_{33}$ are interchanged then the output does not change, i.e. the output $Y = X_{(3)} = X_3$ for all orderings with $X_2, X_5 \leq X_3$ and $X_1, X_4 \geq X_3$. This fact is quite general and is stated formally in Lemma 1 below.

Lemma 1: Let $z_k, 1 \leq k \leq N!$ represent the ordering of the input to a WM filter of size N and weights $\mathbf{w}$. If for this input ordering the output $Y = X_{(i)} = X_j, 1 \leq i, j \leq N$ then the output is the same for all input orderings $z_{\hat{k}}, 1 \leq \hat{k} \leq N!$ obtained by permuting the ranks of the samples within $\mathcal{G}_{ij}$ and $\mathcal{H}_{ij}$.

Proof: For the input ordering $z_k, 1 \leq k \leq N!$ the output $Y = X_{(i)} = X_j, 1 \leq i, j \leq N$ if and only if the following pair of inequalities hold simultaneously.

$$w_j + \sum_{\text{all } n \text{ s.t. } X_n \in \mathcal{G}_{ij}(\mathcal{H}_{ij})} w_n \geq M + 1 \tag{5a}$$

$$\sum_{\text{all } n \text{ s.t. } X_n \in \mathcal{G}_{ij}(\mathcal{H}_{ij})} w_n \leq M \tag{5b}$$

Since the sum of the weights of the samples within $\mathcal{G}_{ij}$ (or $\mathcal{H}_{ij}$) does not change if their ranks are interchanged, any permutation of the same leaves the output unaltered. ∎

We will now state and prove, using Lemma 1, the relation between rank selection probabilities and the output distribution of a WM filter with continuous i.i.d. inputs.

Theorem 1: The output distribution function $F_Y(y)$ and the density function $f_Y(y)$ of a WM filter of size N with i.i.d. inputs is given by

$$F_Y(y) = \sum_{i=1}^{N} P(Y = X_{(i)}) F_i(y) \tag{6a}$$

$$f_Y(y) = \sum_{i=1}^{N} P(Y = X_{(i)}) f_i(y) \tag{6b}$$

where $F_i(y)$ and $f_i(y)$ are the distribution and the density functions respectively, of the i^{th} order statistic for i.i.d. inputs.

Proof: The output distribution $F_Y(y)$ is given by

$$\begin{aligned} F_Y(y) = P(Y \leq y) &= \sum_{i=1}^{N} \sum_{j=1}^{N} P(Y \leq y, Y = X_j, Y = X_{(i)}) \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} P(X_{(i)} \leq y, X_{(i)} = X_j, X_{(i)} = Y) \end{aligned} \tag{7}$$

The last inequality holds because the event $\{Y \leq y, Y = X_j, Y = X_{(i)}\}$ is identical to the event $\{X_{(i)} \leq y, X_{(i)} = X_j, X_{(i)} = Y)\}$ for all values of i and j. Consider the term within the double summation

$$P(X_{(i)} \leq y, X_{(i)} = X_j, X_{(i)} = Y) = \sum_{z_k} P(X_{r_1} \leq \ldots \leq X_{r_{i-1}} \leq X_j \leq X_{r_{i+1}} \leq \ldots \leq X_{r_N}, X_j \leq y, X_j = Y) \tag{8}$$

where the indices $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_N$ are all different with none equal to j. Clearly, $\mathcal{G}_{ij} = \{X_{r_1}, X_{r_2}, \ldots, X_{r_{i-1}}\}$ and $\mathcal{H}_{ij} = \{X_{r_{i+1}}, X_{r_{i+2}}, \ldots, X_{r_N}\}$. From Lemma 1 we know that the output does not change if the ranks of the samples within $\mathcal{G}_{ij}$ and/or $\mathcal{H}_{ij}$ are changed. Thus

$$P(X_{(i)} \leq y, X_{(i)} = X_j, X_{(i)} = Y) = \sum_{\text{distinct sets } \mathcal{G}_{ij} \text{ s.t. } Y = X_{(i)} = X_j} P(X_{(i)} \leq y, \mathcal{G}_{ij}) \tag{9}$$

where $P(X_{(i)} \leq y, \mathcal{G}_{ij})$ is the probability $X_{(i)} \leq y$ for some $\mathcal{G}_{ij}$. Since the input $P(X_{(i)} \leq y, \mathcal{G}_{ij})$ is identical for each distinct $\mathcal{G}_{ij}$ and is given by

$$P(X_{(i)} \leq y, \mathcal{G}_{ij}) = \int_{-\infty}^{y} F_X^{i-1}(\tau)(1 - F_X(\tau))^{N-i} f_X(\tau) d\tau \tag{10}$$

and the summation in (9) boils down to counting all distinct sets $\mathcal{G}_{ij}$ which give $Y = X_{(i)} = X_j$, which has been defined as C_{ij}. Thus

$$\begin{aligned} P(X_{(i)} \le y, X_{(i)} = X_j, X_{(i)} = Y) &= C_{ij} \int_{-\infty}^{y} F_X^{i-1}(\tau)(1 - F_X(\tau))^{N-i} f_X(\tau) d\tau \\ &= \frac{(i-1)!\,(N-i)!\,C_{ij}}{N!} \frac{N!}{(i-1)!\,(N-i)!} F_i(y) \\ &= \frac{P_{ij}}{N!} F_i(y) \\ &= P(Y = X_{(i)}, Y = X_j) F_i(y) \end{aligned} \tag{11}$$

Substituting in (8) and summing over j we get

$$F_Y(y) = \sum_{i=1}^{N} P(Y = X_{(i)}) F_i(y) \tag{12}$$

On diffrentiating with respect to y we get $f_Y(y)$. ∎

The example below illustrates Theorem 1.

Example 1 : The RSP's of a WM filter of size 4 are given by $\mathbf{r} = (0.0, 0.5, 0.5, 0.0)$. For i.i.d. $U(0,1)$ inputs the output distribution is given by

$$F_Y(y) = 0.5(F_2(y) + F_3(y)) \tag{13a}$$

$$f_Y(y) = 0.5(f_2(y) + f_3(y)) \tag{13b}$$

where $F_2(y) = 3y^4 - 8y^3 + 6y^2$, $f_2(y) = 12y(1-y)^2$, $F_3(y) = 4y^3 - 3y^4$, and $f_3(y) = 12y^2(1-y), 0 \le y \le 1$. On substitution this gives $F_Y(y) = 3y^2 - 2y^3$ and $f_Y(y) = 6y(1-y)$. ∎

From (6) it is clear that the output distribution $F_Y(.)$ is a polynomial in the input distribution $F_X(.)$. In the corollary below this fact is used to obtain a simpler expression for $F_Y(.)$.

Corollary 1: The output distribution function $F_Y(y)$ and the density function $f_Y(y)$ of a WM filter of size N with i.i.d. inputs are given by

$$F_Y(y) = \sum_{k=1}^{N} c_k F_X^k(y) \tag{14a}$$

$$f_Y(y) = \sum_{k=1}^{N} k c_k F_X^{k-1}(y) f_X(y) \tag{14b}$$

where,

$$c_k = \sum_{i=1}^{k} \sum_{r=i}^{k} P(Y = X_{(i)}) \binom{N}{r} \binom{N-r}{k-r} (-1)^{k-r} \tag{15}$$

Proof: By substituting equation $(2b)$ in $(7a)$ we get

$$\begin{aligned} F_Y(y) &= \sum_{i=1}^{N} \sum_{r=i}^{N} P(Y = X_{(i)}) \binom{N}{r} F_X^r(y)(1 - F_X(y))^{N-r} \\ &= \sum_{i=1}^{N} \sum_{r=i}^{N} \sum_{m=0}^{N-r} P(Y = X_{(i)}) \binom{N}{r} \binom{N-r}{m} (-1)^{N-r-m} F_X^{N-m}(y) \\ &= \sum_{i=1}^{N} \sum_{r=i}^{N} \sum_{k=r}^{N} P(Y = X_{(i)}) \binom{N}{r} \binom{N-r}{k-r} (-1)^{k-r} F_X^k(y) \end{aligned} \tag{16}$$

On rearranging we get

$$F_Y(y) = \sum_{k=1}^{N}\sum_{i=1}^{k}\sum_{r=i}^{k} P(Y = X_{(i)})\binom{N}{r}\binom{N-r}{k-r}(-1)^{k-r}F_X^k(y) \tag{17}$$

From which (14) and (15) follow. ∎

A broad range of properties of WM filters can be derived using Theorem 1. We will first consider some properties of RSP's itself and their consequences on filter behaviour. In all the cases below the input to the WM filter is assumed i.i.d. and its window size equals N.

Property 1: If a WM filter is an identity filter then $P(Y = X_{(i)}) = 1/N, i = 1, 2, \ldots, N$. ∎

Here identity filter refers to a WM filter whose output is always the same sample in the input window. The index need not be the center of the window. The proof of this property is simple and is therefore omitted. Properties of WM filters which are not identity filters are now examined.

Property 2: For a WM filter which is not an identity filter $P(Y = X_{(1)}) = P(Y = X_{(N)}) = 0$.

Proof: Assume that $P(Y = X_{(1)})$ takes some non-zero value. This would imply that there exists at least one ordering for which the output $Y = X_{(1)}$. Let $Y = X_j$ for this ordering. Then $w_j > \sum_{n=1, n\neq j}^{N} w_n$ which implies that the output is always X_j, i.e. the WM filter is an identity filter. This is a contradiction. Hence $P(Y = X_{(1)}) = 0$. $P(Y = X_{(N)}) = 0$ can be proved similarly. ∎

In the following property where it is shown that the RSP's are distributed symmetrically about the central rank.

Property 3: For any WM filter of size N, $P(Y = X_{(i)}) = P(Y = X_{(N+1-i)}), i = 1, 2, \ldots, N$.

Proof: We have $P(Y = X_{(i)}) =$ (number of orderings for which $Y = X_{(i)})/N!$. Let $z_k, k = 1 \leq k \leq N!$ be an ordering for which $Y = X_j = X_{(i)}$. If the ordering of the samples in z_k is reversed then the output would still be X_j. Since the ordering is reversed the rank of the output will be $N + 1 - i$. Further, an ordering and its reverse form always occur in pairs. Thus the number of orderings for which $Y = X_{(i)}$ and for which $Y = X_{(N+1-i)}$ is the same. Hence $P(Y = X_{(i)}) = P(Y = X_{(N+1-i)})$. ∎

In Table 1 the RSP's and distribution functions of all WM filters of size 5 are listed. The properties stated above can easily be verified from this table. We will now show how Theorem 1 can be used to derive certain characteristics of WM filters. In all the cases below the input is assumed to be i.i.d. with continuous density and distribution functions $f_X(.)$ and $F_X(.)$ respectively. In principle, the results are equally valid for discrete valued inputs.

Property 4: For a WM filter with input distribution $F_X(.)$ if for some $-\infty < t_1, t_2 < \infty$

$$F_X(t_1) = 1 - F_X(t_2) \tag{18a}$$

$$\text{then,} \qquad F_Y(t_1) = 1 - F_Y(t_2) \tag{18b}$$

Proof: For any $F_X(t_1)$ we have $(F_X(t_1) + (1 - F_X(t_1)))^N = 1$. After expanding, transposing, and substituting for $F_X(t_1)$ in this identity we get

$$\begin{aligned}\sum_{r=i}^{N}\binom{N}{r}F_X^r(t_1)(1-F_X(t_1))^{N-r} &= 1 - \sum_{r=0}^{i-1}\binom{N}{N-r}(1-F_X(t_1))^{N-r}F_X^r(t_1) \\ &= 1 - \sum_{r=N+1-i}^{N}\binom{N}{r}F_X^r(t_2)(1-F_X(t_2))^{N-r}\end{aligned} \tag{19}$$

Thus

$$F_i(t_1) = 1 - F_{N+1-i}(t_2) \tag{20}$$

By substituting (20) in (6a) and simplifying we get the stated result. ∎

When the input and output have identical medians then the filter is said to be unbiased in the sense of the median. The following shows that WM filters possess this property.

Weights	C-Matrix	P-Matrix	RSP	SSP	Output Distribution
$(1,1,5,1,1)$	$\begin{pmatrix} 0&0&1&0&0 \\ 0&0&4&0&0 \\ 0&0&6&0&0 \\ 0&0&4&0&0 \\ 0&0&1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&24&0&0 \\ 0&0&24&0&0 \\ 0&0&24&0&0 \\ 0&0&24&0&0 \\ 0&0&24&0&0 \end{pmatrix}$	$(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5})$	$(0,0,1,0,0)$	$F_X(x)$
$(1,3,3,3,1)$	$\begin{pmatrix} 0&0&0&0&0 \\ 0&2&2&2&0 \\ 0&4&4&4&0 \\ 0&2&2&2&0 \\ 0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0 \\ 0&12&12&12&0 \\ 0&16&16&16&0 \\ 0&12&12&12&0 \\ 0&0&0&0&0 \end{pmatrix}$	$(0,\frac{3}{10},\frac{4}{10},\frac{3}{10},0)$	$(0,\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$	$-2F_X^3(x)+3F_X^2(x)$
$(1,2,2,2,4)$	$\begin{pmatrix} 0&0&0&0&0 \\ 0&1&1&1&3 \\ 0&2&2&2&6 \\ 0&1&1&1&3 \\ 0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0 \\ 0&6&6&6&18 \\ 0&8&8&8&24 \\ 0&6&6&6&18 \\ 0&0&0&0&0 \end{pmatrix}$	$(0,\frac{3}{10},\frac{4}{10},\frac{3}{10},0)$	$(0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{3}{6})$	$-2F_X^3(x)+3F_X^2(x)$
$(1,1,1,1,1)$	$\begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 6&6&6&6&6 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 24&24&24&24&24 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$	$(0,0,1,0,0)$	$(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5})$	$6F_X^5(x)-12F_X^4(x)+10F_X^3(x)$
$(1,1,3,1,1)$	$\begin{pmatrix} 0&0&0&0&0 \\ 1&1&4&1&1 \\ 0&0&6&0&0 \\ 1&1&4&1&1 \\ 0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0 \\ 6&6&24&6&6 \\ 0&0&24&0&0 \\ 6&6&24&6&6 \\ 0&0&0&0&0 \end{pmatrix}$	$(0,\frac{2}{5},\frac{1}{5},\frac{2}{5},0)$	$(\frac{1}{10},\frac{1}{10},\frac{6}{10},\frac{1}{10},\frac{1}{10})$	$-2F_X^5(x)+5F_X^4(x)-6F_X^3(x)+4F_X^2(x)$
$(1,2,1,2,1)$	$\begin{pmatrix} 0&0&0&0&0 \\ 0&1&0&1&0 \\ 4&6&4&6&4 \\ 0&1&0&1&0 \\ 0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0 \\ 0&6&0&6&0 \\ 16&24&16&24&16 \\ 0&6&0&6&0 \\ 0&0&0&0&0 \end{pmatrix}$	$(0,\frac{1}{10},\frac{8}{10},\frac{1}{10},0)$	$(\frac{4}{30},\frac{9}{30},\frac{4}{30},\frac{9}{30},\frac{4}{30})$	$4F_X^5(x)-10F_X^4(x)+6F_X^3(x)+F_X^2(x)$
$(1,2,3,3,1)$	$\begin{pmatrix} 0&0&0&0&0 \\ 0&1&2&1&0 \\ 2&4&6&4&2 \\ 0&1&2&1&0 \\ 0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0 \\ 0&6&12&6&0 \\ 8&16&24&16&8 \\ 0&6&12&6&0 \\ 0&0&0&0&0 \end{pmatrix}$	$(0,\frac{1}{5},\frac{3}{5},\frac{1}{5},0)$	$(\frac{2}{30},\frac{7}{30},\frac{12}{30},\frac{7}{30},\frac{2}{30})$	$2F_X^5(x)-5F_X^4(x)+2F_X^3(x)+2F_X^2(x)$

Table 1: C-matrix, P-matrix, and output distributions of WM filters of size 5.
Input is i.i.d.

Property 5: The output of a WM filter is unbiased in the sense of the median.

Proof: Let the median of the input be μ_{MED}. Then $F_X(\mu_{MED}) = 0.5$ or $F_X(\mu_{MED}) = 1 - F_X(\mu_{MED})$. From Property 5 it follows that $F_Y(\mu_{MED}) = 1 - F_Y(\mu_{MED})$ or $F_Y(\mu_{MED}) = 0.5$ which implies that the median of the output is also μ_{MED}. ∎

Recall that for symmetric density functions the median equals the mean. Thus the WM filter is also unbiased in the sense of the mean for symmetric input densities. Furthermore, the output density is also symmetric about the mean.

Once the output density and distribution functions are known then the output moments can be computed in a straightforward manner. In what follows we will examine expected values and moments of the WM filter. As before the input is assumed to be i.i.d. with density $f_X(.)$ and distribution $F_X(.)$ and the window size is N.

Property 6: The n^{th} moment of the output of a WM filter is given by either of the expressions below

$$E(Y^n) = \sum_{i=1}^{N} P(Y = X_{(i)})E(X_{(i)}^n) \tag{21a}$$

$$E(Y^n) = \sum_{k=1}^{N} c_k E(X_{k:k}^n) \tag{21b}$$

where $c_k, k = 1, 2, \ldots, N$ is given by (15) and $E(X_{k:k}^n)$ is the n^{th} moment of the largest order statistic in a window of size k.

Proof: $E(Y^n) = \int_{-\infty}^{\infty} y^n f_Y(y)dy = \sum_{i=1}^{N} P(Y = X_{(i)}) \int_{-\infty}^{\infty} y^n f_i(y)dy = \sum_{i=1}^{N} P(Y = X_{(i)})E(X_{(i)}^n)$. Also, $E(Y^n) = \int_{-\infty}^{\infty} y^n f_Y(y)dy = \sum_{k=1}^{N} c_k \int_{-\infty}^{\infty} ky^n F_X^{k-1}(y) f_X(y)dy$. By definition the integral equals $E(X_{k:k}^n)$. ∎

Statistical properties of the i^{th} order statistic, in particular the largest order statistic, have been studied in great detail in statistical literature.[7], 8 These results can be used to derive properties of WM filters related to its output moments. In the following example we will illustrate Property 6.

Example 2 : Find the expression for the n^{th} moments and evaluate the mean μ_Y and variance σ_Y^2 of the output of a WM filter with weights $\mathbf{w}_0 = (1, 1, 3, 1, 1)$ and i.i.d. inputs with distribution $F_X(x) = x, 0 \le x \le 1$.

From Table 1, the RSP's are $\mathbf{r} = (0, 0.4, 0.2, 0.4, 0)$. The n^{th} order moments of the i^{th} order statistic, $i = 1, 2, 3, 4, 5$ are given as follows: $E(X_{(1)}) = \frac{5}{n+1} - \frac{20}{n+2} + \frac{30}{n+3} - \frac{20}{n+4} + \frac{5}{n+5}$, $E(X_{(2)}) = \frac{20}{n+2} - \frac{60}{n+3} + \frac{60}{n+4} - \frac{20}{n+5}$, $E(X_{(3)}) = \frac{30}{n+3} - \frac{60}{n+4} + \frac{30}{n+5}$, $E(X_{(4)}) = \frac{20}{n+4} - \frac{20}{n+5}$, and $E(X_{(5)}) = \frac{5}{n+5}$. Using (21$a$) we get

$$E(Y^n) = \frac{8}{n+2} - \frac{18}{n+3} + \frac{20}{n+4} - \frac{10}{n+5} \tag{22}$$

Alternately, we have $F_Y(y) = \sum_{k=1}^{5} c_k F_X^k(y)$ where $c_1 = 0, c_2 = 4, c_3 = -6, c_4 = 5$, and $c_5 = -2$. $E(Y^n) = \sum_{k=1}^{5} c_k E(X_{k:k}^n)$ with

$$E(X_{k:k}^n) = \int_0^1 kx^{k-1}x^n dx = \frac{k}{n+k}, \quad k = 1, \ldots, 5 \tag{23}$$

which gives $E(Y^n)$ as in (22) above. From here we get $\mu_Y = 0.5$ and $\sigma_Y^2 = 0.05$. ∎

When the window size N is large ($N \ge 9$), bounds can be placed on the expected value of the order statistics[7] and thus on the output of a WM filter.

Property 7: The mean of the output of a WM filter is bounded by

$$\mu_X - \frac{N-1}{\sqrt{2N-1}}\sigma_X \le E(Y) \le \mu_X + \frac{N-1}{\sqrt{2N-1}}\sigma_X \tag{24}$$

where μ_X and σ_X are the mean and standard deviation repectively, of the input.

Proof: From [7, pp 58] we have

$$E(X_{(N)}) \leq \mu_X + \frac{N-1}{\sqrt{2N-1}}\sigma_X \tag{25a}$$

$$E(X_{(1)}) \geq \mu_X - \frac{N-1}{\sqrt{2N-1}}\sigma_X \tag{25b}$$

i.e. for any $1 \leq i \leq N$ $\mu_X - \frac{N-1}{\sqrt{2N-1}}\sigma_X \leq E(X_{(i)}) \leq \mu_X + \frac{N-1}{\sqrt{2N-1}}\sigma_X$. The result follows after multiplying by $P(Y = X_{(i)})$, each a positive quantity, and adding. ▮

When the window size is large, a close approximation to $E(X_{(i)})$ is given by the value of the i^{th} quantile of the input,[7] i.e.

$$E(X_{(i)}) \approx F_X^{-1}(\frac{i}{N+1}) \tag{26}$$

This implies the following property for WM filters.

Property 8: The mean of the output of a WM filter with window size N is given by

$$E(Y) \approx \sum_{i=1}^{N} P(Y = X_{(i)}) F_X^{-1}(\frac{i}{N+1}) \tag{27}$$

where $F_X(.)$ is the input distribution. ▮

In the foregoing discussion a number of useful properties of WM filters were derived. They are all based on the results in Theorem 1 wherein the key idea was representing the output distribution using rank selection probabilities. In the following sub-section we will examine the counterpart of rank selection probabilities, viz. the sample selection probabilities.

2.3. Sample Selection Probabilities and WM filters

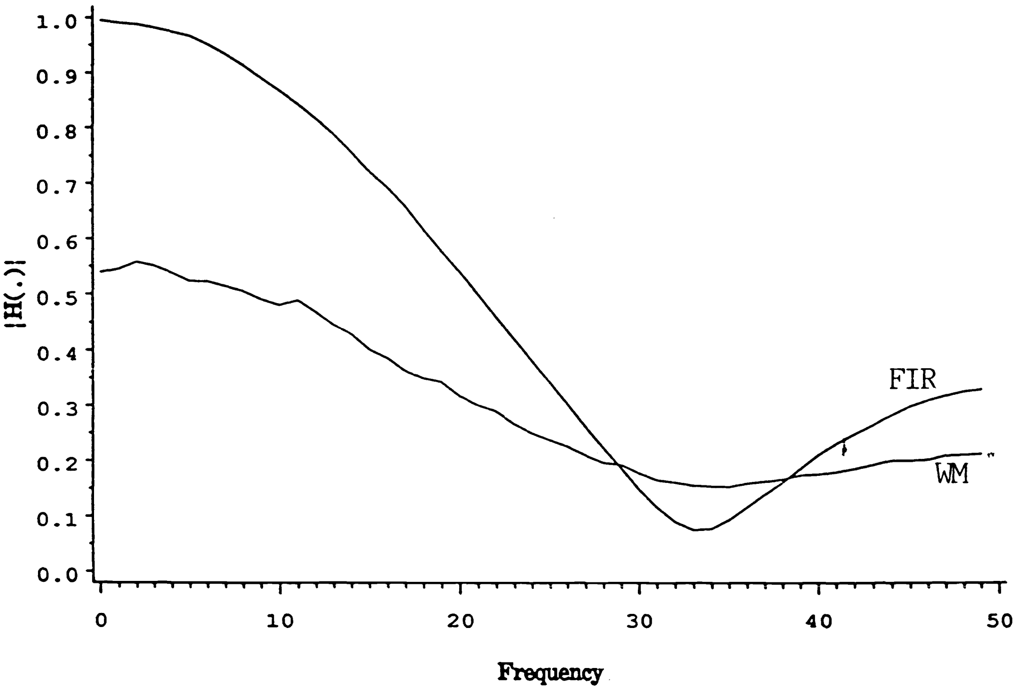
For linear filters with stochastic inputs the output spectrum can be obtained from the autocorrelation function of the input process and the impulse response function of the filter. For non-linear filters such analysis has little meaning because impulse response of a non-linear filter cannot be defined. In an attempt to characterise non-linear filters Mallows[9] hypothesized that a non-linear filter with i.i.d. inputs can be decomposed into a 'linear' and a 'residual' part. The input itself can be decomposed into a sum of processes with Gaussian and non-Gaussian densities respectively. The linear part of the non-linear filter is the linear filter which filters the Gaussian part of the input such that its output is closest to the non-linear filter in the mean-square sense. In simpler but less rigorous terms, the linear part is the linear filter whose outputs for Gaussian inputs is closest to that of non-linear filter in the mean square sense. He also showed that the spectral content of the output of the non-linear filter approximates that of its linear part. Since the frequency response of linear filters is quite well defined, this formulation makes it easy to characterise the output spectrum of a non-linear filter if its linear part is known. In general, finding the linear part of a non-linear filter is rather difficult. However, for filters like the WM filter where the output is always one of the samples from the input window, it was shown that the linear part is a finite impulse response (FIR) filter whose coefficients can be related to the sample selection probabilities. The result is restated for our purposes in Theorem 2 below.

Theorem 2[9]: The linear part of a WM filter of size N is a FIR filter whose coefficients $h_j, j = 1, 2, \ldots, N$ are given by

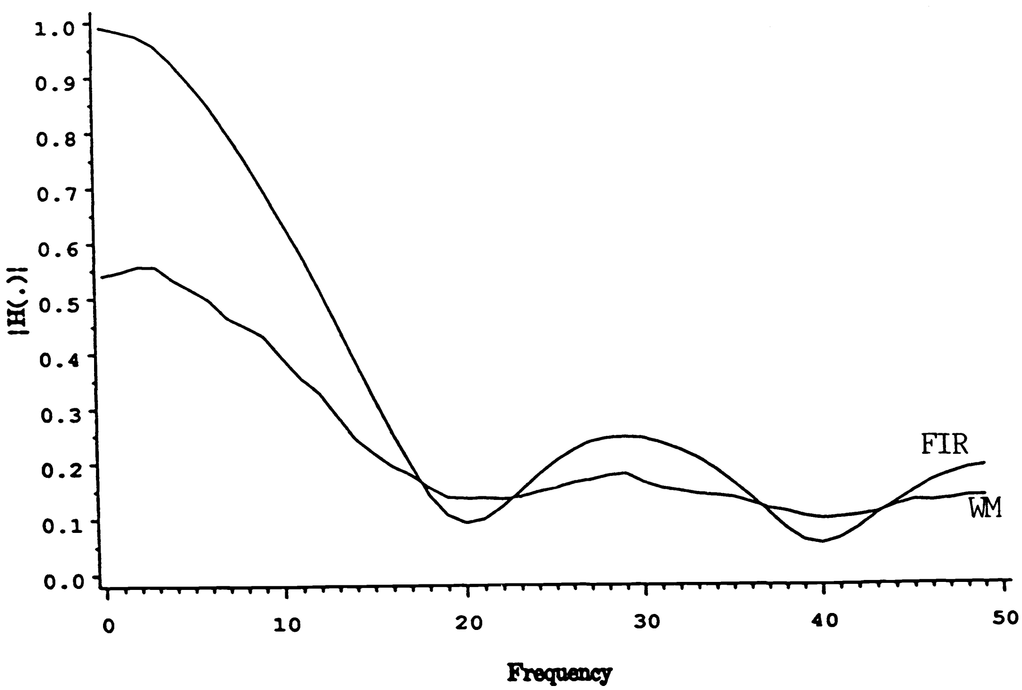
$$h_j = P(Y = X_{N+1-j}). \tag{28}$$

In Table 1 the sample selection probabilities of all WM filters of size 5 are listed. The output spectrum of some of these filters are compared with that of the corresponding FIR filter in Fig. 1a-b. In all the cases the output spectrum closely follows the spectrum of its linear part. Notice that in each case the WM filter shows low-pass characteristics, a fact which is true for all WM filters.

Observation 1: All WM filters with i.i.d. inputs have low-pass characteristics.



a:



b:

Figure 3.1a: Output spectra of a weighted median filter with weights $\mathbf{w} = (1, 3, 3, 3, 1)$ and the corresponding FIR filter. b: Output spectra of a weighted median filter with weights $\mathbf{w} = (1, 1, 1, 1, 1)$ and the corresponding FIR filter.

Proof: Let $h_j, j = 1, 2, \ldots, N$ be the coefficients of the FIR filter which is the linear part of the WM filter. The spectral characteristics of the output of the WM filter is given by those of this FIR filter. Let $H(j\omega)$ denote the Fourier transform of its impulse response function. We have

$$| H(j\omega) |^2 = \sum_{n=1}^{N} h_n^2 + 2 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} h_n h_{n+m} cos\omega m \tag{29}$$

The coefficient of the cosine terms in the equality are always positive since they are probabilities. The observation follows from the fact that for $|w| \le \frac{\pi}{N}$ all terms in (29) will be additive. ∎

3. COMPUTATION OF SELECTION PROBABILITIES

A WM filter is specified either by a Boolean function[10] or by the weights $\mathbf{w}$. Unlike the weights, the Boolean function is a unique representation of a WM filter, but it is convenient and more common to specify a WM filter by its weights. If the selection probabilities are to be of practical use, it should be possible to compute them from either of the two specifications. The problem of evaluating the selection probabilities from the Boolean function is still under study. Here, we will show how the selection probabilities of a WM filter can be obtained from its weights.

3.1. Generating Function and Selection Probabilities

The joint selection probability was defined above as the probability that the output Y of a WM filter equals the i^{th} ranked data and the j^{th} sample simultaneously and is denoted by $P(Y = X_{(i)}, Y = X_j) = P_{ij}/N!, 1 \le i, j \le N$. The rank and sample selection probabilities can be obtained from the joint selection probabilities by summing over the appropriate indices, i.e.

$$P(Y = X_{(i)}) = \sum_{j=1}^{N} \frac{P_{ij}}{N!}, \quad i = 1, 2, \ldots, N \tag{30a}$$

$$P(Y = X_j) = \sum_{i=1}^{N} \frac{P_{ij}}{N!}, \quad j = 1, 2, \ldots, N \tag{30b}$$

Thus the rank and sample selection probabilities can be determined if P_{ij}'s, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$ are known. Recall that $P_{ij} = (i-1)!\ (N-i)!\ C_{ij}$ where C_{ij} is the number of distinct sets $\mathcal{G}_{ij} = \{X_{(1)}, X_{(2)}, \ldots, X_{(i-1)}, X_{(i)} = X_j\}, X_j \notin \mathcal{G}_{ij}$ such that $Y = X(i) = X_j$. The output $Y = X_{(i)} = X_j$ if and only if the following inequalities are true simultaneously

$$w_j + \sum_{n=1}^{i-1} w_{(n)} \ge M + 1 \tag{31a}$$

$$\sum_{n=1}^{i-1} w_{(n)} \le M \tag{31b}$$

where $w_{(n)}, n = 1, 2, \ldots, i-1$ is the weight of $X_{(n)} \in \mathcal{G}_{ij}$ and w_j is the weight of $X_j, j = 1, 2, \ldots, N$. In other words, $Y = X_{(i)} = X_j$ if and only if $M + 1 - w_j \le \sum_{n=1}^{i-1} w_{(n)} \le M$. So the problem of finding C_{ij}, the number of distinct sets $\mathcal{G}_{ij}$, can be reformulated as follows: What is the total number of ways of choosing exactly $i-1$ elements from the set of integers $\mathcal{A} = \{w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_N\}$ such that their sum is no less than $M + 1 - w_j$ and no greater than M? This can be obtained by summing the number of partitions $p(n)$ over $n = M + 1 - w_j, M + 2 - w_j, \ldots, M$ with the condition that each partition consists of exactly $i-1$ summands chosen from $\mathcal{A}$.[11] Two weights may have identical values but they are treated as distinct because they are associated with different samples and therefore may correspond to distinct sets $\mathcal{G}_{ij}$. C_{ij} can then be evaluated as shown in Theorem 3 below.

Theorem 3: Let $U_j(a, x)$ be a polynomial in arbitrary variables a, x given by

$$U_j(a, x) = \prod_{n=1, n \ne j}^{N} (1 + a x^{w_n}) \tag{32}$$

and let coefficient of a^{i-1} in $U_j(a,x)$, $V_{ij}(x)$ be given by

$$V_{ij}(x) = \sum_{n=0}^{2M+1-w_j} b_n x^n \tag{33}$$

$b_n \geq 0, n = 0, \ldots, 2M+1-w_j$. Then

$$C_{ij} = \sum_{n=M+1-w_j}^{M} b_n \tag{34}$$

Proof: From [11,pp.194-195] we know that the number of partitions of n into exactly $i-1$ distinct summands taken from among the elements in $\mathcal{A} = \{w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_N\}$ is the coefficient of $a^{i-1}x^n$ in $U_j(a,x)$. C_{ij} is the sum of the coefficients of $a^{i-1}x^n, n = M+1-w_j, M+2-w_j, \ldots, M$. Thus $C_{ij} = \sum_{n=M+1-W_j}^{M} b_n$, where $b_n,\ n = 0,\ 2M+1-w_j$ are the coefficients of x^n in $V_{ij(x)}$. ∎

Once C_{ij} is known P_{ij} can be computed from which the selection probabilities can be obtained as in (30). The example below illustrates how Theorem 3 is used to evaluate the rank and sample selection probabilities.

Example 3: A WM filter of size 3 has weights $\mathbf{w} = (7,6,2)$. Find its rank and sample selection probabilities.

Here $w_1 + w_2 + w_3 = 15$, or $M = 7$ where $\sum_{n=1}^{3} w_n = 2M+1$. The generating function $U_j(a,x), j = 1,2,3$ are given as follows: $U_1(a,x) = (1+ax^6)(1+ax^2) = 1 + a(x^2+x^6) + a^2x^8; U_2(a,x) = (1+ax^7)(1+ax^2) = 1 + a(x^2+x^7) + a^2x^9; U_3(a,x) = (1+ax^6)(1+ax^7) = 1 + a(x^6+x^7) + a^2x^{13}$. From which we get the polynomials $V_{ij}(x), i = 1,2,3, j = 1,2,3$: $V_{11}(x) = 1, V_{21}(x) = x^2 + x^6, V_{31}(x) = x^8$; $V_{12}(x) = 1, V_{22}(x) = x^2 + x^7, V_{32}(x) = x^9$; $V_{13}(x) = 1, V_{23}(x) = x^6 + x^7, V_{33}(x) = x^{13}$. This gives: $C_{11} = 0, C_{21} = 0, C_{31} = 0$; $C_{12} = 2, C_{22} = 2, C_{32} = 2$; $C_{13} = 0, C_{23} = 0, C_{33} = 0$. The rank and sample selection probabilities are given by $\mathbf{r} = (0,1,0)$ and $\mathbf{s} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ respectively. ∎

A convenient way of writing the elements P_{ij} and C_{ij} is $1 \leq i,j \leq N$ is to represent them in matrix form. Thus $\mathbf{P} = \{P_{ij}\}_{N\times N}$ and $\mathbf{C} = \{C_{ij}\}_{N\times N}$. For obvious reasons $\mathbf{P}$ is called the *permutation* or *P-matrix* and $\mathbf{C}$ is called the *combination* or *C-matrix*. The following observation relating the selection probabilities and the P-matrix follows from (30).

Observation 2: The i^{th} rank selection probability, $P(Y = X_{(i)}), i = 1,2,\ldots,N$ is given by $P(Y = X_{(i)}) =$ (i^{th} row sum of the P-matrix)$/N!$ and the j^{th} rank selection probability, $P(Y = X_{(j)}), j = 1,2,\ldots,N$, is given by $P(Y = X_j) =$ (j^{th} column sum of the P-matrix)$/N!$. ∎

Example 3 (continued): The C-matrix and P-matrix are given by

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

respectively. ∎

In Table 1 the combination and permutation matrices of some WM filters is given. Notice that in each of the cases $P_{ij} = P_{N+1-i,j}$.

Using the combination and permutation matrices is a convenient way of representing the joint selection probabilities. The joint selection probabilities can be related to the rank and sample selection probabilities therefore it is worthwhile to investigate the properties of $\mathbf{C}$ and $\mathbf{P}$ matrices and examine the consequences on the rank and sample selection probabilities. Consider two WM filters with weights $\mathbf{w} = (w_1, w_2, \ldots, w_N)$ and $\hat{\mathbf{w}} = (\hat{w}_1, \hat{w}_2, \ldots \hat{w}_N)$ such that $w_j = \hat{w}_j$, for all $j \neq j', j'', w_{j'} = \hat{w}_{j''}, w_{j''} = \hat{w}_{j'}$. In other words, $\hat{\mathbf{w}}$ is obtained by exchanging the weights at the positions j' and j''. Let the C-matrix and P-matrix corresponding to $\mathbf{w}$ and $\hat{\mathbf{w}}$ be $\mathbf{C}, \mathbf{P}$ and $\hat{\mathbf{C}}, \hat{\mathbf{P}}$ respectively. The following property indicates the relation between these pairs of matrices.

Property 9: If the weights $w_{j'}$ and $w_{j''}, 1 \leq j', j'' \leq N$, of a WM filter with weights $\mathbf{w}$, C-matrix $\mathbf{C}$ and P-matrix $\mathbf{P}$ are exchanged to give another WM filter with weights $\hat{\mathbf{w}}$, C-matrix $\hat{\mathbf{C}}$ and P-matrix $\hat{\mathbf{P}}$ then $\hat{\mathbf{C}}$ ($\hat{\mathbf{P}}$) can be obtained from $\mathbf{C}$ ($\mathbf{P}$) by exchanging the columns at j' and j'' of $\mathbf{C}$ ($\mathbf{P}$).

Proof: Let the generating functions of the two filters be $U_j(.,.)$ and $\hat{U}_j(.,.), j = 1,2,\ldots,N$ respectively. Since $w_{j'}$ and $w_{j''}$ are exchanged, $U_{j'}(.,.) = \hat{U}_{j''}, j \neq j', j''$, $U_{j'}(.,.) = \hat{U}_{j''}$, and $U_{j''}(.,.) = \hat{U}_{j'}$. Which implies that $V_{ij'}(.,.) = \hat{V}_{ij''}, j \neq j', j''$, $V_{ij'}(.,.) = \hat{V}_{ij''}$, and $V_{ij''}(.,.) = V_{ij'}$ for each $i = 1,2,\ldots,N$. Hence the elements of the C-matrix, $\mathbf{C}$ and $\hat{\mathbf{C}}$ are such that $C_{ij'}(.,.) = \hat{C}_{ij''}, j \neq j', j''$, $C_{ij'}(.,.) = \hat{C}_{ij''}$, and $C_{ij''}(.,.) = C_{ij'}$ for each $i = 1,2,\ldots,N$. Thus $\hat{\mathbf{C}}$ can be obtained from $\mathbf{C}$ by exchanging the j' and j'' columns. The proof is completed by noting that $P_{ij} = (i-1)!\,(N-i)!\,C_{ij}$. ∎

As a result of Property 9 the following conclusions can be drawn about the rank and sample selection probabilities.

Property 10: For any permutation of the weights $w_1, w_2, \ldots, w_N$ the RSP's remain unchanged and the SSP's are permuted in the same manner as the weights.

Proof: Any permutation of weights can be considered to be a series of exchanges, two weights at time. Thus any permutation of the weights is an identical permutation of the columns of the P-matrix which affects only the order of the elements in each row. Since $P(Y = X_{(i)}) = (i^{th} \text{ row sum})/N!$ it follows that the RSP's remain unchanged. And, since $P(Y = X_j) = (j^{th} \text{ column sum})/N!$, it follows that the SSP's will be permuted in the same manner as the weights. ∎

For example, the WM filters $\mathbf{w}_1 = (1,2,3,2,1)$, $\mathbf{w}_2 = (1,2,1,2,3)$, and $\mathbf{w}_3 = (1,1,2,2,3)$ will have the same output distributions $F_Y(y) = 2F_X^5(y) - 5F_X^4(y) + 2F_X^3(y) + 2F_X^2(y)$ when the input is i.i.d. with distribution $F_X(.)$. The SSP's are given by $\mathbf{s}_1 = (0.1, 0.2, 0.4, 0.2, 0.1)$, and $\mathbf{s}_2 = (0.1, 0.2, 0.1, 0.2, 0.4)$, and $\mathbf{s}_3 = (0.1, 0.1, 0.2, 0.2, 0.4)$ respectively. Thus $\mathbf{s}_2$ and $\mathbf{s}_3$ could have been obtained from $\mathbf{s}_1$ merely by exchanging the appropriate SSP's. Given any WM filter then, one needs to compute the selection probabilities for only one arrangement of weights.

4. CONCLUSIONS

In this chapter, rank and sample selection probabilities have been introduced as a novel method for statistical analysis of WM filters. Rank selection probabilities simplified formulation of the output distribution and related statistics and sample selection probabilities gave an idea of the approximate frequency response of the non-linear filter. Properties of selection probabilities also indicated their importance in understanding the behaviour of WM filters.

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