Capacity Bounds for Alternating Two-Path Relay Channels

Woohyuk Chang, Sae-Young Chung, and Yong H. Lee

***Abstract*— We propose a decode-and-forward (DF) strategy for a relay channel with two half-duplex relays that alternately forward their received messages. The proposed strategy is to combine two distinctive schemes which differently treat a codeword that each relay receives from the other relay. One lets each relay consider it as interference, while the other makes each relay decode and forward it to the destination. We show the channel conditions under which each scheme outperforms the other and combining both schemes is better than simply choosing the best of them. Even though we can verify that the proposed DF strategy achieves a cut-set upper bound only in certain cases, we numerically show that the gap between them is relatively small or negligible for most cases.**

I. Introduction

The relay is a fundamental building block to make up communication networks. A first comprehensive study on the simplest three-terminal relay channel was done by Cover and El Gamal in [1], where a relay node is assumed to communicate in a full-duplex manner such that it transmits and receives at the same time. Since then, various relay channels have been investigated including half-duplex relays for practical applications [2]–[7] and multiple relays for improving spectral efficiency [8]–[13]. However, the theoretical performance limits on such relay channels are still unknown except in certain cases.

In this paper, we consider a relay channel with two half-duplex relays that alternately forward their received messages such that one of the relays transmits data to the destination while the other receives data from the source. (It is assumed that there is no direct link between the source and destination.) This alternating two-path relay channel is first introduced in [13] where both an amplifying-and-forward (AF) and a decode-and-forward (DF) strategies are proposed and their performances are examined.

The objective of this paper is to develop an alternative DF strategy which outperforms the previous AF and DF strategies in [13]. In particular, we present a DF strategy which combines two distinctive schemes in dealing with a codeword that each relay receives from the other relay. In one scheme, each relay considers it as interference where the dirty-paper coding (DPC) [14] is applied since the interference from the other relay can be pre-subtracted at the source (the source already knows what the other relay is going to transmit). In the other scheme, each relay decodes the codeword received from the other relay and forwards it to the destination, i.e., each relay acts as a relay for the other relay as well as the source. Since each relay delivers the codeword to the destination directly as well as through the other relay, the block-Markov coding [1], [15], [16] can be effectively used. We show when each scheme outperforms the other. Combining both schemes is also shown to be better than simply choosing the best of them. Even though we can verify that the proposed DF strategy achieves a cut-set upper bound only in certain cases, we numerically show that the gap between them is relatively small or negligible for most cases.

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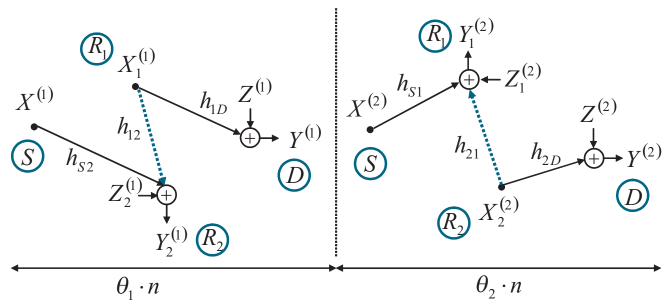

Fig. 1. Gaussian alternating two-path relay channel.

The rest of the paper is organized as follows. In Section II, we introduce a Gaussian alternating two-path relay channel. Section III derives an upper bound on its capacity and develops the proposed DF strategy. Section IV provides some numerical examples. In Section V, we conclude our paper.

II. Gaussian Alternating Two-Path Relay Channels

We consider a Gaussian relay channel with two half-duplex relays that alternately transmit over two consecutive blocks during n channel uses as shown in Fig. 1. In phase 1 corresponding to the first block of length $\theta_1 n$ with $0 \leq \theta_1 \leq 1$, the source and relay 1 transmit while relay 2 and the destination receive. On the other hand, in phase 2 corresponding to the second block of length $\theta_2 n$ with $\theta_2 = 1 - \theta_1$, the source and relay 2 transmit while relay 1 and the destination receive. These two phases are repeated at every two blocks. For convenience, $\theta_1 n$ is assumed to be an integer. It is also assumed that there is no direct path from the source to the destination due to shadowing or a long distance between them.

In phase $i \in \{1, 2\}$, relay j and the destination receive

$$y_j^{(i)} = h_{Sj} x^{(i)} + h_{ij} x_i^{(i)} + z_j^{(i)}, \tag{1}$$

$$y^{(i)} = h_{iD} x_i^{(i)} + z^{(i)}, \tag{2}$$

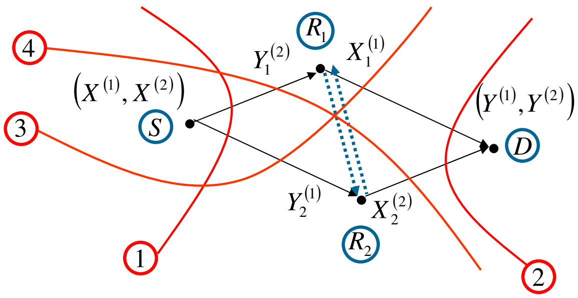

Fig. 2. Two cut-sets.

respectively, where $j = 3 - i$, $x^{(i)}$ and $x_i^{(i)}$ are the channel inputs at the source and relay i in phase i, respectively, $z_j^{(i)} \sim \mathcal{N}(0,1)$, $z^{(i)} \sim \mathcal{N}(0,1)$ are independent of each other, and h_{Sj}, h_{ij} and h_{iD} are real-valued constant channel gains between the corresponding nodes. We assume that the source transmits $x^{(i)}$ with power $P_S^{(i)}$ under an average power constraint P_S such that $\theta_1 P_S^{(1)} + \theta_2 P_S^{(2)} = P_S$. Likewise, relay i for $i \in \{1,2\}$ is assumed to transmit $x_i^{(i)}$ with power P_i under an average power constraint of $\theta_i P^{(i)} = P_i$.

III. Capacity Bounds

In this section, we consider upper and lower bounds on the capacity of the Gaussian alternating two-path relay channel given in Section II. The upper bound can be easily found from the well known max-flow min-cut theorem [17]. Applying four cut-sets in Fig. 2, we get the following upper bound.

Theorem 1 (Upper Bound): The capacity for the Gaussian alternating two-path relay channel is upper bounded by

$$C^+ = \max_{\substack{0 \le \theta_1, \rho_1, \rho_2 \le 1 \\ 0 \le P_S^{(1)} \le \frac{P_S}{\theta_1}}} \min \left\{ C_1^+, C_2^+, C_3^+, C_4^+ \right\} \tag{3}$$

where

$$C_1^+ = \theta_1 C \left(g_{S2} P_S^{(1)} \left(1 - \rho_1^2\right)\right) + \theta_2 C \left(g_{S1} P_S^{(2)} \left(1 - \rho_2^2\right)\right), \tag{4}$$

$$C_2^+ = \theta_1 C \left(g_{1D} P_1^{(1)} \right) + \theta_2 C \left(g_{2D} P_2^{(2)} \right), \tag{5}$$

$$C_3^+ = \theta_1 C \Big(g_{S2} P_S^{(1)} + g_{1D} P_1^{(1)} + \left(1 - \rho_1^2\right) g_{S2} g_{1D} P_S^{(1)} P_1^{(1)} + 2\rho_1 \sqrt{g_{S2} g_{12} P_S^{(1)} P_1^{(1)}} \Big), \tag{6}$$

$$C_4^+ = \theta_2 C \Big(g_{S1} P_S^{(2)} + g_{2D} P_2^{(2)} + \left(1 - \rho_2^2\right) g_{S1} g_{2D} P_S^{(2)} P_2^{(2)} + 2\rho_2 \sqrt{g_{S1} g_{21} P_S^{(2)} P_2^{(2)}} \Big), \tag{7}$$

where $C(x) = \frac{1}{2}\log_2(1+x)$ [bits/1D] and $g_x = |h_x|^2$.

Proof: For the first cut-set, applying Theorem 14.10.1 in [17],

$$\begin{aligned} nR &\le \theta_1 I \left(X^{(1)}; Y_2^{(1)}, Y^{(1)} | X_1^{(1)} \right) + \theta_2 I \left(X^{(2)}; Y_1^{(2)}, Y^{(2)} | X_2^{(2)} \right) \\ &= \theta_1 I \left(X^{(1)}; Y_2^{(1)} | X_1^{(1)} \right) + \theta_2 I \left(X^{(2)}; Y_1^{(2)} | X_2^{(2)} \right), \\ &= \theta_1 C \left(g_{S2} \mathbb{E}\big[|X^{(1)}|^2\big] \left(1 - \rho_1^2\right)\right) + \theta_2 C \left(g_{S1} \mathbb{E}\big[|X^{(2)}|^2\big] \left(1 - \rho_2^2\right)\right) \\ &\le \theta_1 C \left(g_{S2} P_S^{(1)} \left(1 - \rho_1^2\right)\right) + \theta_2 C \left(g_{S1} P_S^{(2)} \left(1 - \rho_2^2\right)\right) \end{aligned} \tag{8}$$

where for $i \in \{1,2\}$,

$$\rho_i = \frac{\mathbb{E}\big[X^{(i)} X_i^{(i)}\big]}{\sqrt{\mathbb{E}\big[|X^{(i)}|^2\big] \mathbb{E}\big[|X_i^{(i)}|^2\big]}}. \tag{9}$$

Likewise, from the second cut-set,

$$\begin{aligned} nR &\le \theta_1 I \left(X^{(1)}, X_1^{(1)}; Y^{(1)} \right) + \theta_2 I \left(X^{(2)}, X_2^{(2)}; Y^{(2)} \right) \\ &= \theta_1 I \left(X_1^{(1)}; Y^{(1)} \right) + \theta_2 I \left(X_2^{(2)}; Y^{(2)} \right) \\ &= \theta_1 C \left(g_{1D} \mathbb{E}\big[|X_1^{(1)}|^2\big] \right) + \theta_2 C \left(g_{2D} \mathbb{E}\big[|X_2^{(2)}|^2\big] \right) \\ &\le \theta_1 C \left(g_{1D} P_1^{(1)} \right) + \theta_2 C \left(g_{2D} P_2^{(2)} \right). \end{aligned} \tag{10}$$

From the third cut-set,

$$\begin{aligned} nR &\le \theta_1 I \left(X^{(1)}, X_1^{(1)}; Y_2^{(1)}, Y^{(1)} \right) \\ &= \theta_1 I \left(X^{(1)}, X_1^{(1)}; Y_2^{(1)} \right) + \theta_1 I \left(X^{(1)}, X_1^{(1)}; Y^{(1)} | Y_2^{(1)} \right) \\ &= \theta_1 I \left(X^{(1)}, X_1^{(1)}; Y_2^{(1)} \right) + \theta_1 I \left(X_1^{(1)}; Y^{(1)} | Y_2^{(1)} \right) \\ &\quad + \theta_1 I \left(X^{(1)}; Y^{(1)} | X_1^{(1)}, Y_2^{(1)} \right) \\ &= \theta_1 H \left(Y_2^{(1)} \right) - \theta_1 H \left(Y_2^{(1)} | X^{(1)}, X_1^{(1)} \right) \\ &\quad + \theta_1 H \left(Y^{(1)} | Y_2^{(2)} \right) - \theta_1 H \left(Y^{(1)} | X^{(1)}, X_1^{(1)}, Y_2^{(2)} \right) \\ &\quad - \theta_1 H \left(Y^{(1)} | X^{(1)}, X_1^{(1)}, Y_2^{(2)} \right) \\ &= \theta_1 C \Big(g_{S2} \mathbb{E}\big[|X^{(1)}|^2\big] + g_{1D} \mathbb{E}\big[|X_1^{(1)}|^2\big] \\ &\qquad + \left(1 - \rho_1^2\right) g_{S2} g_{1D} \mathbb{E}\big[|X^{(1)}|^2\big] \mathbb{E}\big[|X_1^{(1)}|^2\big] \\ &\qquad + 2\rho_1 \sqrt{g_{S2} g_{12} \mathbb{E}\big[|X^{(1)}|^2\big] \mathbb{E}\big[|X_1^{(1)}|^2\big]} \Big) \end{aligned} \tag{11}$$

which is maximized when $\mathbb{E}\big[|X^{(1)}|^2\big] = P_S^{(1)}$ and $\mathbb{E}\big[|X_1^{(1)}|^2\big] = P_1^{(1)}$, and hence we have (6). Likewise, we can get (7) from the fourth cut-set. This concludes the proof. ■

We next consider an achievable scheme. Suppose that the alternating two-path relay channel is operating in phase 1, where the source and relay 1 transmit, and relay 2 and the destination receive. When relay 2 decodes the received signal, there can be two different schemes in dealing with a codeword received from relay 1. One is to consider it as interference, and hence relay 2 decodes and forwards only a codeword received from the source. A notable thing is such interference is already known by the source since it was transmitted at the source in the previous block. In this case, the DPC [14] can be used effectively to make each relay decode the codeword from the source as if there is

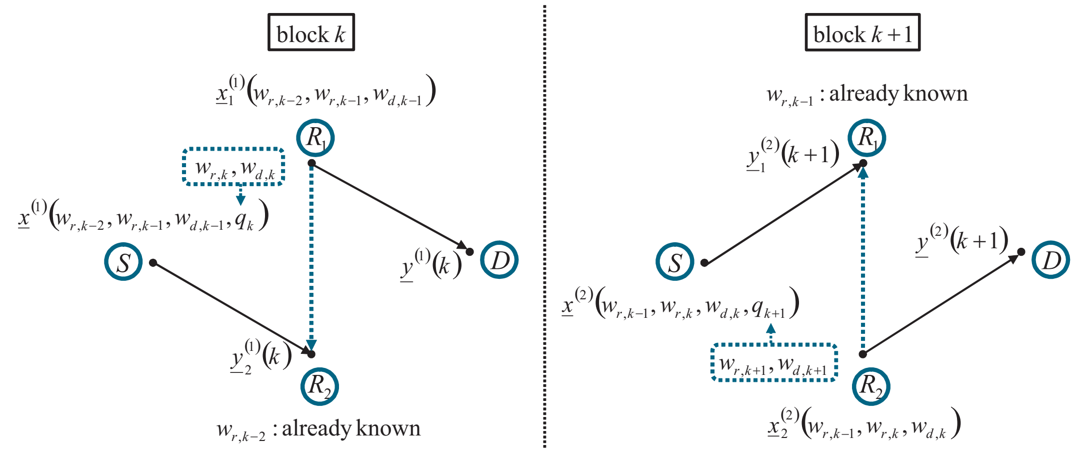

Fig. 3. Two consecutive blocks in the alternating two-path relay channel assuming that block k is in phase 1.

no interference from the other relay. Hence, two half-duplex relays act as one full-duplex relay with no interference between them.

The other scheme is to consider the codeword received from relay 1 as one to be forwarded to the destination, and hence relay 2 needs to decode it. In this case, relay 2 acts as a relay for both the source and relay 1 such that it forwards the codewords received from the source and relay 1 to relay 1 and the destination, respectively. We can see how this scheme can be effective through the following example. Suppose that $g_{S1} = g_{2D}$ are much bigger than $g_{S2} = g_{1D}$. In this case, a significant part of the received message from the source at relay 1 cannot be forwarded to the destination utilizing only the channel g_{1D}. However it can be delivered to the destination through relay 2. Moreover, since relay 2 has a worse channel g_{S2} through which it receives only a small amount of the message from the source, it can afford to forward the message from relay 1.

In this paper, we propose a DF strategy which combines both schemes. Encoding proceeds as follows. We assume that blocks k and $k+1$ are in phases 1 and 2, respectively, as shown in Fig. 3. The message w_k to be sent at the source in block k is split into $w_{r,k}$ and $w_{d,k}$ that are the messages to be forwarded to relay 1 and the destination by relay 2 in the next block $k+1$, respectively. At this time, relay 1 forwards three messages $w_{r,k-2}$, $w_{r,k-1}$ and $w_{d,k-1}$ among which $w_{r,k-2}$ is received from relay 2, and $w_{r,k-1}$ and $w_{d,k-1}$ are received from the source in the previous block $k-1$. Since relay 1 forwards $w_{r,k-1}$ to relay 2 while $w_{r,k-2}$ and $w_{d,k-1}$ are forwarded to the destination, $w_{r,k-2}$ and $w_{d,k-1}$ can be seen as interference at relay 2. However, $w_{r,k-2}$ is already known by relay 2, and hence only $w_{d,k-1}$ actually acts as interference at relay 2. When the source encodes the message tuple $(w_{r,k}, w_{d,k})$ into an integer q_k in the manner of the DPC, $w_{d,k-1}$ acts as a noncausally known channel interference. Note that we use a regular block-Markov encoding [15], [16] to encode $\underline{x}_1^{(1)}$ and $\underline{x}_2^{(2)}$ where codebooks for $\underline{x}_1^{(1)}$ and $\underline{x}_2^{(2)}$ are statistically independent. Codewords that are sent in the first four blocks are shown in Table I.

Decoding proceeds as follows. In the end of block k, relay 2 decodes both $w_{r,k-1}$ and q_k from its channel output in block k. Then, relay 2 restores $(w_{r,k}, w_{d,k})$ which is the bin index that q_k belongs to. Note that although $(w_{r,k}, w_{d,k})$ is decoded as if $w_{d,k-1}$ does not exist by the DPC, $w_{r,k-1}$ is still decoded in the presence of $w_{d,k-1}$ as interference. In the end of block $k+1$, the destination decodes $w_{r,k-1}$ by using sliding-window decoding [15], [16] from its channel outputs in both blocks k and $k+1$. Then, the destination decodes $w_{d,k-1}$ from its channel output in block k.

The achievable rate for the proposed DF strategy is given by the following theorem.

Theorem 2 (Achievable Rate: Decode-and-Forward): The capacity for the Gaussian alternating two-path relay channel is lower bounded by

$$C_{DF}^{-} = \max_{\underline{R} \in \mathcal{R}_{DF}} R_r^{(1)} + R_d^{(1)} + R_r^{(2)} + R_d^{(2)}, \qquad (12)$$

where

$$\mathcal{R}_{DF} = \bigcup_{\substack{0 \le \theta_1, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \le 1 \\ 0 \le P_S^{(1)} \le \frac{P_S}{\theta_1}}}$$

$$\Big\{ \underline{R} = \left(R_r^{(1)}, R_d^{(1)}, R^{(1)\prime}, R_r^{(2)}, R_d^{(2)}, R^{(2)\prime} \right) :$$

$$R^{(1)\prime} < \theta_1 C\left(g_{S2}\alpha_1 P_S^{(1)} + \frac{g_{S2}\alpha_1 P_S^{(1)} g_{12}\bar{\beta}_1\bar{\gamma}_1 P_1^{(1)}}{1+g_{S2}\alpha_1 P_S^{(1)}} \right) \triangleq M_1,$$

$$R^{(1)\prime} + R_r^{(2)} < M_1$$
$$+\theta_1 C\left(\frac{g_{S2}\bar{\alpha}_1 P_S^{(1)} + g_{12}\bar{\beta}_1\gamma_1 P_1^{(1)} + 2\sqrt{g_{S2}g_{12}\bar{\alpha}_1\bar{\beta}_1\gamma_1 P_S^{(1)} P_1^{(1)}}}{1+g_{S2}\alpha_1 P_S^{(1)} + g_{12}\bar{\beta}_1\bar{\gamma}_1 P_1^{(1)}} \right),$$

$$R_r^{(1)} + R_d^{(1)} < R^{(1)\prime} - \theta_1 C\left(\frac{g_{S2}\alpha_1 P_S^{(1)} g_{12}\bar{\beta}_1\bar{\gamma}_1 P_1^{(1)}}{\left(1+g_{S2}\alpha_1 P_S^{(1)}\right)^2} \right),$$

$$R_r^{(1)} < \theta_1 C\left(\frac{g_{1D}\beta_1 P_1^{(1)}}{1+g_{1D}\bar{\beta}_1 P_1^{(1)}} \right) + \theta_2 C\left(\frac{g_{2D}\bar{\beta}_2\gamma_2 P_2^{(2)}}{1+g_{2D}\bar{\beta}_2\bar{\gamma}_2 P_2^{(2)}} \right),$$

$$R_d^{(1)} < \theta_2 C\left(g_{2D}\bar{\beta}_2\bar{\gamma}_2 P_2^{(2)} \right),$$

$$R^{(2)\prime} < \theta_2 C\left(g_{S1}\alpha_2 P_S^{(2)} + \frac{g_{S1}\alpha_2 P_S^{(2)} g_{21}\bar{\beta}_2\bar{\gamma}_2 P_2^{(2)}}{1+g_{S1}\alpha_2 P_S^{(2)}} \right) \triangleq M_2,$$

TABLE I

CODEWORDS BEING SENT IN THE FIRST FOUR BLOCKS TO ACHIEVE THE RATE C_{DF}^{-}

block 1	block 2	block 3	block 4
$\underline{x}_{11}^{(1)}(1)$ $\underline{x}_{12}^{(1)}(1)$ $\underline{x}_{1}^{(1)}(1,1,1)$ $\underline{u}^{(1)}(1,1,q_1)$ $\underline{x}^{(1)}(1,1,1,q_1)$	$\underline{x}_{21}^{(2)}(1)$ $\underline{x}_{22}^{(2)}(1,w_{r,1})$ $\underline{x}_{2}^{(2)}(1,w_{r,1},w_{d,1})$ $\underline{u}^{(2)}(1,w_{r,1},q_2)$ $\underline{x}^{(2)}(1,w_{r,1},w_{d,1},q_2)$	$\underline{x}_{11}^{(1)}(w_{r,1})$ $\underline{x}_{12}^{(1)}(w_{r,1},w_{r,2})$ $\underline{x}_{1}^{(1)}(w_{r,1},w_{r,2},w_{d,2})$ $\underline{u}^{(1)}(w_{r,1},w_{r,2},q_3)$ $\underline{x}^{(1)}(w_{r,1},w_{r,2},w_{d,2},q_3)$	$\underline{x}_{21}^{(2)}(w_{r,2})$ $\underline{x}_{22}^{(2)}(w_{r,2},w_{r,3})$ $\underline{x}_{2}^{(2)}(w_{r,2},w_{r,3},w_{d,3})$ $\underline{u}^{(2)}(w_{r,2},w_{r,3},q_4)$ $\underline{x}^{(2)}(w_{r,2},w_{r,3},w_{d,3},q_4)$

$$
\begin{aligned}
&R^{(2)\prime} + R_r^{(1)} < M_2 \\
&\quad +\theta_2 C\left(\frac{g_{S1}\bar{\alpha}_2 P_S^{(2)} + g_{21}\bar{\beta}_2\gamma_2 P_2^{(2)} + 2\sqrt{g_{S1}g_{21}\bar{\alpha}_2\bar{\beta}_2\gamma_2 P_S^{(2)} P_2^{(2)}}}{1+g_{S1}\alpha_2 P_S^{(2)} + g_{21}\bar{\beta}_2\bar{\gamma}_2 P_2^{(2)}}\right), \\
&R_r^{(2)} + R_d^{(2)} < R^{(2)\prime} - \theta_2 C\left(\frac{g_{S1}\alpha_2 P_S^{(2)} g_{21}\bar{\beta}_2\bar{\gamma}_2 P_2^{(2)}}{\left(1+g_{S1}\alpha_2 P_S^{(2)}\right)^2}\right), \\
&R_r^{(2)} < \theta_2 C\left(\frac{g_{2D}\beta_2 P_2^{(2)}}{1+g_{2D}\bar{\beta}_2 P_2^{(2)}}\right) + \theta_1 C\left(\frac{g_{1D}\bar{\beta}_1\gamma_1 P_1^{(1)}}{1+g_{1D}\bar{\beta}_1\bar{\gamma}_1 P_1^{(1)}}\right), \\
&R_d^{(2)} < \theta_1 C\left(g_{1D}\bar{\beta}_1\bar{\gamma}_1 P_1^{(1)}\right)\Big\}
\end{aligned}
\tag{13}
$$

and $\bar{x} = 1 - x$.

Proof: When we construct a codeword, its length is determined by the superscript such that superscripts (1) and (2) correspond to codewords of length $\theta_1 n$ and $\theta_2 n$, respectively. For example, $\underline{x}^{(1)} = \left(x_1^{(1)}, \cdots, x_{\theta_1 n}^{(1)}\right)$ and $\underline{x}^{(2)} = \left(x_1^{(2)}, \cdots, x_{\theta_2 n}^{(2)}\right)$.

Random Code Construction:

1) For each $a \in \{1,2\}$, choose $2^{nR_r^{(a)}}$ i.i.d. $\underline{x}_{a1}^{(a)}$ with probability $\prod_i p\big(x_{a1,i}^{(a)}\big)$.
   Label these $\underline{x}_{a1}^{(a)}(w_r^{(a)})$, $w_r^{(a)} \in \left[1, 2^{nR_r^{(a)}}\right]$.
2) For each $\underline{x}_{a1}^{(a)}(w_r^{(a)})$ and $b = 3-a$, choose $2^{nR_r^{(b)}}$ i.i.d. $\underline{x}_{a2}^{(a)}$ with probability $\prod_i p\big(x_{a2,i}^{(a)} \big| x_{a1,i}^{(a)}(w_r^{(a)})\big)$.
   Label these $\underline{x}_{a2}^{(a)}(w_r^{(a)}, w_r^{(b)})$, $w_r^{(b)} \in \left[1, 2^{nR_r^{(b)}}\right]$.
3) For each $\underline{x}_{a1}^{(a)}(w_r^{(a)})$ and $\underline{x}_{a2}^{(a)}(w_r^{(a)}, w_r^{(b)})$, choose $2^{nR_d^{(b)}}$ i.i.d. $\underline{x}_{a}^{(a)}$ with probability $\prod_i p\big(x_{a,i}^{(a)} \big| x_{a1,i}^{(a)}(w_r^{(a)}),\, x_{a2,i}^{(a)}(w_r^{(a)}, w_r^{(b)})\big)$.
   Label these $\underline{x}_{a}^{(a)}(w_r^{(a)}, w_r^{(b)}, w_d^{(b)})$, $w_d^{(b)} \in \left[1, 2^{nR_d^{(b)}}\right]$.
4) For each $\underline{x}_{a1}^{(a)}(w_r^{(a)})$ and $\underline{x}_{a2}^{(a)}(w_r^{(a)}, w_r^{(b)})$, choose $2^{nR^{(a)\prime}}$ i.i.d. $\underline{u}^{(a)}$ with probability
$$\prod_i p\big(u_i^{(a)} \big| x_{a1,i}^{(a)}(w_r^{(a)}), x_{a2,i}^{(a)}(w_r^{(a)}, w_r^{(b)})\big).$$
   Label these $\underline{u}^{(a)}(w_r^{(a)}, w_r^{(b)}, q^{(a)})$, $q^{(a)} \in \left[1, 2^{nR^{(a)\prime}}\right]$.
5) For each a, randomly partition the set $\left\{1, \cdots, 2^{nR^{(a)\prime}}\right\}$ into $2^{n(R_r^{(a)}+R_d^{(a)})}$ bins $S_{s_r^{(a)}, s_d^{(a)}}$ with $(s_r^{(a)}, s_d^{(a)}) \in \left[1, 2^{nR_r^{(a)}}\right] \times \left[1, 2^{nR_d^{(a)}}\right]$.
6) For each $w_r^{(a)}, w_r^{(b)}, w_d^{(b)}$ and $q^{(a)}$, choose i.i.d. $\underline{x}^{(a)}$ with probability
$$\prod_i p\big(x^{(a)} \big| x_{a1,i}^{(a)}(w_r^{(a)}), x_{a2,i}^{(a)}(w_r^{(a)}, w_r^{(b)}), x_{a,i}^{(a)}(w_r^{(a)}, w_r^{(b)}, w_d^{(b)}), u_i^{(a)}(w_r^{(a)}, w_r^{(b)}, q^{(a)})\big).$$
   Label this vector $\underline{x}^{(a)}(w_r^{(a)}, w_r^{(b)}, w_d^{(b)}, q^{(a)})$.

We consider K blocks where each odd and even blocks consist of $\theta_1 n$ and $\theta_2 n$ symbols, respectively. A sequence of $K-2$ message tuples $(w_{r,k}, w_{d,k}) \in \left[1, 2^{nR_r^{(1)}}\right] \times \left[1, 2^{nR_d^{(1)}}\right]$ for $k = 1, 3, \cdots, K-3$ and $(w_{r,k}, w_{d,k}) \in \left[1, 2^{nR_r^{(2)}}\right] \times \left[1, 2^{nR_d^{(2)}}\right]$ for $k = 2, 4, \cdots, K-2$ will be sent over the channel in nK transmissions. As $K \to \infty$, for fixed n, the rate $R(K-2)/K$ is arbitrarily close to R with $R = R_r^{(1)} + R_d^{(1)} + R_r^{(2)} + R_d^{(2)}$.

Encoding:

1) Denoting $b = 3 - a$, let $(w_{r,k-1}, w_{d,k-1}) \in \left[1, 2^{nR_r^{(b)}}\right] \times \left[1, 2^{nR_d^{(b)}}\right]$ and $(w_{r,k}, w_{d,k}) \in \left[1, 2^{nR_r^{(a)}}\right] \times \left[1, 2^{nR_d^{(a)}}\right]$ be new messages to be sent in blocks $k-1$ and k at the source, respectively, where $w_{r,k-1}$ and $w_{r,k}$ are to be delivered by utilizing both relays while $w_{d,k-1}$ and $w_{d,k}$ are to be delivered by utilizing only one of them. Set $(w_{r,-1}, w_{r,0}, w_{d,0}) = (1,1,1)$.
   In the beginning of block k, the source finds $q_k \in S_{w_{r,k}, w_{d,k}}$ such that $\underline{u}^{(a)}(w_{r,k-2}, w_{r,k-1}, q_k)$ is jointly typical with $\underline{x}_{a1}^{(a)}(w_{r,k-2})$, $\underline{x}_{a2}^{(a)}(w_{r,k-2}, w_{r,k-1})$ and $\underline{x}_{a}^{(a)}(w_{r,k-2}, w_{r,k-1}, w_{d,k-1})$. There is at least one such q_k if
$$R^{(a)\prime} - (R_r^{(a)} + R_d^{(a)}) > \theta_a I\Big(U^{(a)}; X_a^{(a)} \Big| X_{a1}^{(a)}, X_{a2}^{(a)}\Big). \tag{14}$$
   Then, the source sends $\underline{x}^{(a)}(w_{r,k-2}, w_{r,k-1}, w_{d,k-1}, q_k)$.
2) In the beginning of block $k+1$, relay b already knows $(w_{r,k-1}, w_{r,k}, w_{d,k})$ from decoding step 1) and sends $\underline{x}_b^{(b)}(w_{r,k-1}, w_{r,k}, w_{d,k})$.

Decoding:

1) In the end of block k, assuming $w_{r,k-2}$ was correctly decoded in the past, relay b decodes $(w_{r,k-1}, q_k)$ if and only if there exists an unique $(w_{r,k-1}, q_k)$ such that $\big(\underline{x}_{a1}^{(a)}(w_{r,k-2}), \underline{x}_{a2}^{(a)}(w_{r,k-2}, w_{r,k-1}), \underline{u}^{(a)}(w_{r,k-2}, w_{r,k-1}, q_k), \underline{y}_b^{(a)}(k)\big)$ are jointly typical. This can be done reliably if
$$R^{(a)\prime} < \theta_a I\left(U^{(a)}; Y_b^{(a)} \Big| X_{a1}^{(a)}, X_{a2}^{(a)}\right), \tag{15}$$
$$R^{(a)\prime} + R_r^{(b)} < \theta_a I\left(U^{(a)}, X_{a2}^{(a)}; Y_b^{(a)} \Big| X_{a1}^{(a)}\right). \tag{16}$$
   Then, relay b knows $(w_{r,k}, w_{d,k})$ which is the bin index that q_k belongs to, i.e., $q_k \in S_{w_{r,k}, w_{d,k}}$.

2) In the end of block $k+1$, assuming $w_{r,k-2}$ was correctly decoded in the past, the destination decodes $w_{r,k-1}$ if and only if there exists an unique $w_{r,k-1}$ such that $\left(\underline{x}_{a2}^{(a)}(w_{r,k-2},w_{r,k-1}),\underline{x}_{b1}^{(b)}(w_{r,k-1}),\underline{y}^{(a)}(k),\underline{y}^{(b)}(k+1)\right)$ are jointly typical. This can be done reliably if

$$R_r^{(b)} < \theta_b I\left(X_{b1}^{(b)};Y^{(b)}\right)+\theta_a I\left(X_{a2}^{(a)};Y^{(a)}\middle|X_{a1}^{(a)}\right) \quad (17)$$

by using the sliding-window decoding [15], [16].
Then, the destination decodes $w_{d,k-1}$ if and only if there exists an unique $w_{d,k-1}$ such that $\left(\underline{x}_{a1}^{(a)}(w_{r,k-2}),\right.$ $\underline{x}_{a2}^{(a)}(w_{r,k-2},w_{r,k-1}),\underline{x}_{a}^{(a)}(w_{r,k-2},w_{r,k-1},w_{d,k-1}),$ $\left.\underline{y}^{(a)}(k)\right)$ are jointly typical. This can be done reliably if

$$R_d^{(b)} < \theta_a I\left(X_{a}^{(a)};Y^{(a)}\middle|X_{a1}^{(a)},X_{a2}^{(a)}\right). \quad (18)$$

Let $X_{a1}^{(a)} \sim \mathcal{N}\left(0,\beta_a P_a^{(a)}\right)$, $Q^{(a)} \sim \mathcal{N}\left(0,\bar{\beta}_a\gamma_a P_a^{(a)}\right)$, $S^{(a)} \sim \mathcal{N}\left(0,\bar{\beta}_a\bar{\gamma}_a P_a^{(a)}\right)$, and $V^{(a)} \sim \mathcal{N}\left(0,\alpha_a P_S^{(a)}\right)$, $a\in\{1,2\}$, which are independent of each other. After letting $X_{a2}^{(a)}=X_{a1}^{(a)}+Q^{(a)}$, $X_{a}^{(a)}=X_{a2}^{(a)}+S^{(a)}$, $U_a^{(a)}=V^{(a)}+\frac{g_{Sb}\alpha_a P_S^{(a)}}{1+g_{Sb}\alpha_a P_S^{(a)}}S^{(a)}$ and $X^{(a)}=V^{(a)}+\sqrt{\frac{\bar{\alpha}_a P_S^{(a)}}{\bar{\beta}_a\gamma_a P_a^{(a)}}}X_{a2}^{(a)}$, and applying them into (14)–(18), we can get Theorem 2. ■

The following corollaries provide two special cases of the proposed DF strategy. They correspond to the previously mentioned two distinctive schemes which differently treat a codeword that one relay receives from the other relay.

Corollary 1 (Achievable Rate: Dirty-Paper Coding): The capacity for the Gaussian alternating two-path relay channel is lower bounded by

$$C_{DP}^{-}=\max_{\underline{R}\in\mathcal{R}_{DP}} R_d^{(1)}+R_d^{(2)}, \quad (19)$$

where

$$\mathcal{R}_{DP}=\bigcup_{\substack{0\le\theta_1\le1\\0\le P_S^{(1)}\le\frac{P_S}{\theta_1}}}\left\{\underline{R}=\left(R_d^{(1)},R_d^{(2)}\right):\right.$$
$$R_d^{(1)}<\theta_1 C\left(g_{S2}P_S^{(1)}\right), R_d^{(1)}<\theta_2 C\left(g_{2D}P_2^{(2)}\right),$$
$$\left.R_d^{(2)}<\theta_2 C\left(g_{S1}P_S^{(2)}\right), R_d^{(2)}<\theta_1 C\left(g_{1D}P_1^{(1)}\right)\right\} \quad (20)$$

Proof: C_{DP}^{-} is obtained by substituting $\alpha_1=\alpha_2=1$, $\beta_1=\beta_2=\gamma_1=\gamma_2=0$ and $R_r^{(1)}=R_r^{(2)}=0$ to C_{DF}^{-} in Theorem 2. ■

Corollary 2 (Achievable Rate: Block-Markov Coding): The capacity for the Gaussian alternating two-path relay channel is lower bounded by

$$C_{BM}^{-}=\max_{\underline{R}\in\mathcal{R}_{BM}} R_r^{(1)}+R_r^{(2)}, \quad (21)$$

where

$$\mathcal{R}_{BM}=\bigcup_{\substack{0\le\theta_1,\alpha_1,\alpha_2,\beta_1,\beta_2\le1\\0\le P_S^{(1)}\le\frac{P_S}{\theta_1}}}\left\{\underline{R}=\left(R_r^{(1)},R_r^{(2)}\right):\right.$$
$$R_r^{(1)}<\theta_1 C\left(g_{S2}\alpha_1 P_S^{(1)}\right),$$
$$R_r^{(1)}+R_r^{(2)}<\theta_1 C\left(g_{S2}P_S^{(1)}+g_{12}\bar{\beta}_1P_1^{(1)}+2\sqrt{g_{S2}g_{12}\bar{\alpha}_1\bar{\beta}_1P_S^{(1)}P_1^{(1)}}\right),$$
$$R_r^{(1)}<\theta_1 C\left(\frac{g_{1D}\beta_1P_1^{(1)}}{1+g_{1D}\bar{\beta}_1P_1^{(1)}}\right)+\theta_2 C\left(g_{2D}\bar{\beta}_2P_2^{(2)}\right),$$
$$R_r^{(2)}<\theta_2 C\left(g_{S1}\alpha_2 P_S^{(2)}\right),$$
$$R_r^{(1)}+R_r^{(2)}<\theta_2 C\left(g_{S1}P_S^{(2)}+g_{21}\bar{\beta}_2P_2^{(2)}+2\sqrt{g_{S1}g_{21}\bar{\alpha}_2\bar{\beta}_2P_S^{(2)}P_2^{(2)}}\right),$$
$$\left.R_r^{(2)}<\theta_2 C\left(\frac{g_{2D}\beta_2P_2^{(2)}}{1+g_{2D}\bar{\beta}_2P_2^{(2)}}\right)+\theta_1 C\left(g_{1D}\bar{\beta}_1P_1^{(1)}\right)\right\} \quad (22)$$

Proof: C_{BM}^{-} is obtained by substituting $\gamma_1=\gamma_2=1$, $R_r^{(1)}=R^{(1)\prime}=0$, $R_r^{(2)}=R^{(2)\prime}=0$ and $R_d^{(1)}=R_d^{(2)}=0$ to C_{DF}^{-} in Theorem 2. ■

Note that while C_{DP}^{-} is determined regardless of g_{12} and g_{21}, they should be large enough for C_{BM}^{-} so that the transmitted codeword at each relay is fully decoded at the other relay as well as the destination. This also happens in the three-terminal relay channels in [1] where the link between the source and the relay should be good enough for the relay to fully decode the codeword received from the source.

In some cases, we can show that the proposed DF strategy achieves the upper bound. We first consider a looser upper bound given as follows. Since C_1^{+} in (4) is maximized when $\rho_1=\rho_2=0$,

$$C^{+}\le C_{12}^{+}\triangleq\max_{\substack{0\le\theta_1\le1\\0\le P_S^{(1)}\le\frac{P_S}{\theta_1}}}\min\left\{C_1^{+}|_{\rho_1=\rho_2=0},C_2^{+}\right\}. \quad (23)$$

Although we consider θ_1 and $P_S^{(1)}$ as variables to be optimized throughout this paper, we first assume them fixed. Then, we can easily verify that C_{DP}^{-} achieves the upper bound C_{12}^{+} for the following two cases.

1a) $\theta_1 C\left(g_{S2}P_S^{(1)}\right)\ge\theta_2 C\left(g_{2D}P_2^{(2)}\right)$ and $\theta_2 C\left(g_{S1}P_S^{(2)}\right)\ge\theta_1 C\left(g_{1D}P_1^{(1)}\right)$

1b) $\theta_1 C\left(g_{S2}P_S^{(1)}\right)<\theta_2 C\left(g_{2D}P_2^{(2)}\right)$ and $\theta_2 C\left(g_{S1}P_S^{(2)}\right)<\theta_1 C\left(g_{1D}P_1^{(1)}\right)$

C_{BM}^{-} is also easily shown to achieve the upper bound C_{12}^{+} with $\alpha_1^*=\alpha_2^*=1$ and $\beta_1^*=\beta_2^*=0$ if g_{12} and g_{21} are large enough to satisfy one of the following conditions.

2a) $\theta_2 C\left(g_{2D}P_2^{(2)}\right)+\theta_1 C\left(g_{1D}P_1^{(1)}\right)\le M$ and 1a)

2b) $\theta_1 C\left(g_{S2}P_S^{(1)}\right)+\theta_2 C\left(g_{S1}P_S^{(2)}\right)\le M$ and 1b)

where

$$M=\min\left\{\theta_1 C\left(g_{S2}P_S^{(1)}+g_{12}P_1^{(1)}\right),\theta_2 C\left(g_{S1}P_S^{(2)}+g_{21}P_2^{(2)}\right)\right\}. \quad (24)$$

If θ_1 and $P_S^{(1)}$ are considered as variables to be optimized, it is not clear when the proposed DF strategy achieves the upper bound. However, when the channel gains are symmetric such as $g_{S1} = g_{S2}$, $g_{1D} = g_{2D}$ and $P_1 = P_2$, we can easily verify it. Since the channel gains are symmetric, the optimal $\theta_1^* = \frac{1}{2}$ and $P_S^{(1)*} = P_S$ for all C_{12}^+, C_{DP}^- and C_{BM}^-. Once θ_1^* and $P_S^{(1)*}$ are determined, verifying conditions for achieving C_{12}^+ is the same as the above argument of assuming them fixed.

We provide another simple example where conditions for achieving C_{12}^+ can be easily verified. Suppose that $g_{S1} = g_{2D} = 0$ and $g_{21} \to \infty$. Note that the optimal $\theta_1^* = 1$ in both C_{12}^+ and C_{BM}^- while $C_{DP}^- = 0$ regardless of θ_1^*. Then, (23) reduces to

$$C_{12}^+ = \max_{P_S^{(1)}} \min \left\{ C\left(g_{S2} P_S^{(1)}\right), C\left(g_{1D} P_1^{(1)}\right) \right\}, \quad (25)$$

which can be achieved by C_{BM}^- with $\alpha_1^* = \beta_1^* = 1$ regardless of α_2^* and β_2^* since as $g_{21} \to \infty$, it is possible to deliver the information from relay 2 to relay 1 in a very short time ($\theta_2^* = 0$). Similarly, C_{BM}^- achieves C_{12}^+ when $g_{S2} = g_{1D} = 0$ and $g_{12} \to \infty$.

IV. Numerical Examples

The capacity bounds are shown in Fig. 4 for various $g_{12} = g_{21}$ when the channel gains are symmetric such that $P_S = P_1 = P_2 = 10$ and $g_{S1} = g_{S2} = g_{1D} = g_{2D} = 1$. Since C_{DF}^- results from the most general strategy in our considerations, it should be larger than or equal to both C_{DP}^- and C_{BM}^-. As already explained, C_{DP}^- achieves the upper bound C^+ regardless of g_{12} and g_{21}, and hence $C_{DF}^- = C_{DP}^-$. On the other hand, C_{BM}^- achieves it asymptotically only when $g_{12} = g_{21}$ are large enough. Since when $g_{12} = g_{21}$ are not large enough, the transmission rate of each relay is limited so that its transmitted codeword could be decoded by the other relay, C_{BM}^- decreases as $g_{12} = g_{21} \to 0$.

For comparison, we also show the achievable rates C_{AF}^- and C_{RW}^- of the AF and DF strategies proposed in [13], respectively. When we compute C_{AF}^-, it is assumed that a sequence of 30 messages are transmitted at the source and the all preciously transmitted codewords are perfectly canceled when the destination decodes the current codeword. Whenever each relay amplifies the received signal and forwards it to the other relay through the path h_{12} or h_{21}, the noise is also amplified and eventually accumulated at each relay. Hence, C_{AF}^- decreases as h_{12} or h_{21} increases. Since when C_{RW}^- is computed in [13], each relay does successively decoding or decoding in the presence of the interference from the other relay, $C_{RW}^- \leq C_{DP}^-$. However, C_{RW}^- achieves C^+ asymptotically in Fig. 4 when $g_{12} = g_{21}$ are large enough as well as pretty small.

Fig. 5 shows the capacity bounds for various $g_{12} = g_{21}$ when the channel gains are asymmetric such that $P_S = P_1 = P_2 = 10$, $g_{S1} = g_{2D} = 10$ and $g_{S2} = g_{1D} = 1$. We can see that C_{DF}^- asymptotically achieves C^+ or the gap between C^+ and C_{DF}^- is relatively small. Note that C_{DF}^- is strictly larger than both C_{DP}^- and C_{BM}^- around $10 \sim 20$ dB, and it explains why C_{DF}^- is needed instead of simply choosing the better one between C_{DP}^- and C_{BM}^-. While C_{DP}^- is not affected by $g_{12} = g_{21}$, C_{BM}^- increases as $g_{12} = g_{21} \to \infty$ since the path $h_{S1} \to h_{12} \to h_{2D}$ is more actively utilized.

In Fig. 6, the capacity bounds for various $g_{S1} = g_{2D}$ are shown when $P_S = P_1 = P_2 = 10$, $g_{S2} = g_{1D} = g_{12} = g_{21} = 1$. Although we can only verify that $C_{DF}^- = C_{DP}^-$ achieves C^+ when the channel gains are symmetric such that $g_{S1} = g_{2D} = 1$, numerical results show that the gap between C^+ and C_{DF}^- is relatively small or negligible in most cases. When $g_{S1} = g_{2D}$ is relatively small, C_{BM}^- is larger than C_{DP}^- since g_{12} is relatively large enough for relay 2 to easily decode the codeword received from relay 1. However, as $g_{S1} = g_{2D}$ increase, the transmission rate of relay 1 is limited so that its transmitted codeword could be decoded at relay 2 even after experiencing the relatively small g_{12}, and hence C_{BM}^- is smaller than C_{DP}^-. Finally, we can see $C_{RW}^- \leq C_{DP}^-$ and $C_{AF}^- \leq C_{DF}^-$ in all three figures.

V. Conclusion

For the Gaussian alternating two-path relay channels, we propose a DF strategy to combine two distinctive schemes which differently treat a codeword that each relay receives from the other relay. We show that combining both schemes can be better than simply choosing the best of them. Under the symmetric channel condition, we also show that it is optimal for each relay to consider the codeword from the other relay as interference. Finally, we can see that the gap between the achievable rate of the proposed DF strategy and the upper bound is relatively small or negligible in most cases.

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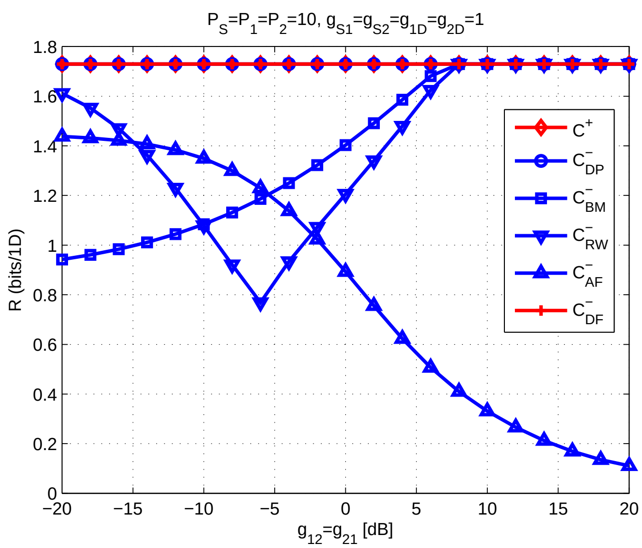

Fig. 4. Numerical results for $P_S = P_1 = P_2 = 10$, $g_{S1} = g_{S2} = g_{1D} = g_{2D} = 1$.

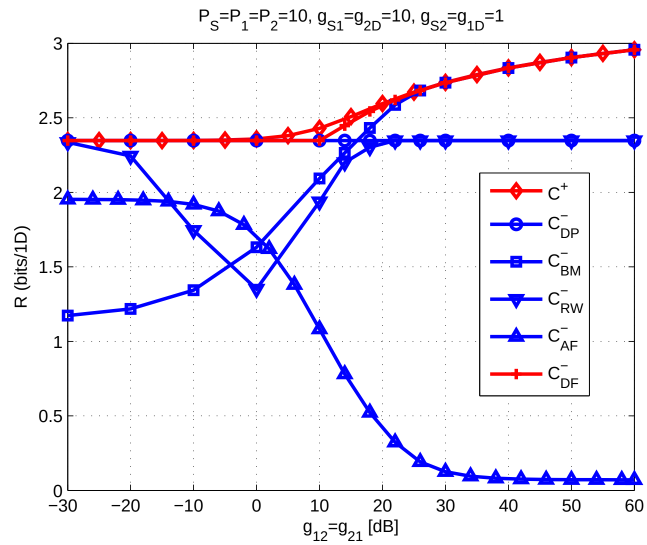

Fig. 5. Numerical results for $P_S = P_1 = P_2 = 10$, $g_{S1} = g_{2D} = 10$, $g_{S2} = g_{1D} = 1$.

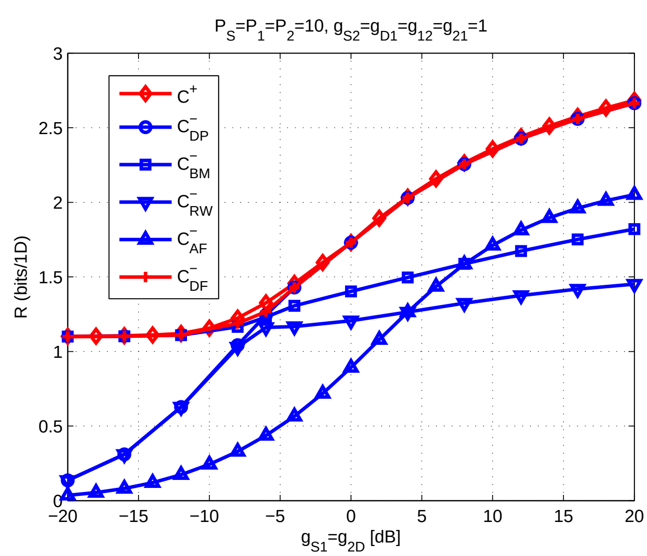

Fig. 6. Numerical results for $P_S = P_1 = P_2 = 10$, $g_{S2} = g_{1D} = g_{12} = g_{21} = 1$.