Capacity Bounds for Two-Way Relay Channels

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***Abstract*—We provide achievable rate regions for two-way relay channels (TRC). At first, for a binary TRC, we show that the subspace-sharing of linear codes can achieve the capacity region. And, for a Gaussian TRC, we propose the subset-sharing of lattice codes. In some cases, the proposed lattice coding scheme can achieve within $1/2$-bit the capacity and is asymptotically optimal at high signal-to-noise ratio (SNR) regimes.**

I. Introduction

We consider two-way relay channels (TRC) as shown in Fig. 1. Nodes 1 and 2 want to exchange messages with each other only through a relay node. There is no direct communication link between them, and all nodes operate in full-duplex mode. We are interested in the capacity region of the TRC. Although the capacity region of the general TRC is still unknown, several schemes have been proposed and their achievable rate region were studied. [1]-[6]. Especially, as the network coding gathers much interest in these days, communication schemes for the TRC are revisited in the context of network coding for wireless networks [5]-[8].

In this work, we introduce some capacity bounds for TRC's. In particular, for a binary TRC, we show that the subspace-sharing of linear codes can achieve the capacity region. For a Gaussian TRC, we propose the subset-set sharing of lattice codes. We also show that, in some cases, the proposed scheme achieves the upper bound of capacity region within $1/2$-bit, and is asymptotically optimal at high SNR regimes.

II. System Model and Some Basic Bounds

We consider a two-way relay channel with three full-duplex nodes as Fig. 1. The variables of the channel are:

- $W_i \in \{1, \ldots, 2^{nR_i}\}$: Messages of node i,
- $\mathbf{X}_i = [X_{i1}, \ldots, X_{in}]^T$: Codewords of node i,
- $\mathbf{Y}_R = [Y_{R1}, \ldots, Y_{Rn}]^T$: Channel output at the relay,
- $\mathbf{X}_R = [X_{R1}, \ldots, X_{Rn}]^T$: Codewords of the relay,
- $\mathbf{Y}_i = [Y_{i1}, \ldots, Y_{in}]^T$: Channel outputs at node i,
- $\hat{W}_i \in \{1, \ldots, 2^{nR_i}\}$: Message estimates at node i,

where $i = 1, 2$ and R_i is the information rate of node i, $i = 1, 2$. Node i transmits its symbol X_{ik} to the relay through the memoryless uplink channel specified by $p(y_R|x_1, x_2)$. The transmit symbol X_{ik} is generally a function of message W_i and past channel outputs $Y_i^{k-1} = Y_{i1}, \ldots, Y_{ik-1}$, i.e., $X_{ik} = f_{ik}\left(W_i, Y_i^{k-1}\right)$. At the same time, the relay transmits symbol X_{Rk} to nodes 1 and 2 through the memoryless downlink channel specified by $p(x_1, x_2|y_R)$. Since the relay

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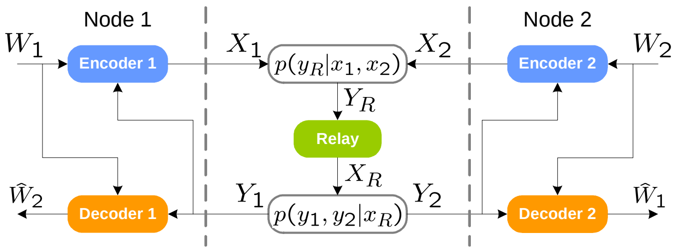

Fig. 1. Two-way relay channel

has no messages of its own, X_{Rk} is formed as a function of past channel outputs $Y_R^{k-1} = Y_{R1}, \ldots, Y_{Rk-1}$, i.e., $X_{Rk} = f_{Rk}\left(Y_R^{k-1}\right)$. At node 1, the message estimate $\hat{W}_2$ is computed as a function of its past channel outputs and its message, i.e., $\hat{W}_2 = g_1(W_1, \mathbf{Y}_1)$. The decoding of node 2 follows similarly.

Now, we introduce some well-known bounds of the capacity region for the general TRC.

A) Outer bound (cut-set bound): Based on the cut-set bound [15], we can compute the outer bound of the capacity region for the TRC. The cut-set bound is computed as follows [1]:

$$\begin{aligned} R_1 &\le \min\{I(X_1; Y_R|X_2), I(X_R; Y_2)\}, \\ R_2 &\le \min\{I(X_2; Y_R|X_1), I(X_R; Y_1)\}. \end{aligned} \tag{1}$$

B) Inner bound (decode-and-forward): As we see in Fig 1, the uplink of the TRC is a multiple-access channel. Thus we can consider the scheme that the relay decodes both of the messages from nodes 1 and 2. The downlink of TRC is basically a broadcast channel except that the decoders have side-information to exploit [9], namely their own messages transmitted in the uplink.

This scheme is called decode-and-forward (DF) relaying and the achievable rate region is studied in [2] for half-duplex cases. However, it can be directly extended to the full-duplex cases as

$$R_1 \le \min\{I(X_1; Y_R|X_2), I(X_R, Y_2)\}, \tag{2a}$$

$$R_2 \le \min\{I(X_2; Y_R|X_1), I(X_R, Y_1)\}, \tag{2b}$$

$$R_1 + R_2 \le I(X_1, X_2; Y_R). \tag{2c}$$

Note that (2a) and (2b) are the same as (1). However, there is another limitation to the sum rate (2c), which is sometimes called the multiplexing loss [1], [2].

III. Binary TRC

As a specific case of the TRC, we first consider a binary TRC, where all links are composed of memoryless binary symmetric channels (BSC). In this case, the received signal at the relay is given by

$$\mathbf{y}_R = \mathbf{x}_1 \oplus \mathbf{x}_2 \oplus \mathbf{z}_R, \tag{3}$$

where $\oplus$ denotes binary addition, and $\mathbf{z}_R$ is the error vector whose components are i.i.d. Bernoulli random variables with $\Pr\{Z_{Rk}=1\} = \epsilon_R$, $k = 1,\ldots,n$. Similarly, the signal received by node i, $i=1$ or 2, is given by

$$\mathbf{y}_i = \mathbf{x}_R \oplus \mathbf{z}_i, \tag{4}$$

where $\mathbf{z}_i$ is the error vector whose components are i.i.d. Bernoulli random variables with $\Pr\{Z_{ik}=1\} = \epsilon_i$, $k = 1,\ldots,n$. Using the result of Section II, we can compute the outer bound of the capacity region for the binary TRC as follows [1]:

$$\begin{aligned} R_1 &\le \min\{1-H(\epsilon_R), 1-H(\epsilon_2)\}, \\ R_2 &\le \min\{1-H(\epsilon_R), 1-H(\epsilon_1)\}. \end{aligned} \tag{5}$$

In [1], it was shown that this outer bound can really be achieved. The technique used in [1] for the uplink is the linear coding with time sharing. This can be seen as a special case of the subspace-sharing scheme, which will be explained below. If $R_1 \ge R_2$, we define binary linear codes (BLC) for node 1 and 2 as follows:

$$\begin{aligned} \mathcal{C}_1 &\triangleq \Bigg\{\mathbf{x}_1 | \mathbf{x}_1 = \begin{bmatrix}\mathbf{u}_{11} & \mathbf{u}_{12}\end{bmatrix}\begin{bmatrix}\mathbf{G}_{11} \\ \mathbf{G}_{12}\end{bmatrix}, \\ &\qquad \mathbf{u}_{11} \in \{0,1\}^{n(R_1-R_2)}, \mathbf{u}_{12} \in \{0,1\}^{nR_2}\Bigg\}, \\ \mathcal{C}_2 &\triangleq \left\{\mathbf{x}_2 | \mathbf{x}_2 = \mathbf{u}_2\mathbf{G}_{12}, \mathbf{u}_{12} \in \{0,1\}^{nR_2}\right\}, \end{aligned} \tag{6}$$

where $\mathbf{G}_{11}$ and $\mathbf{G}_{12}$ are $n(R_1-R_2)\times n$ and $nR_2 \times n$ binary matrices, respectively. It is well known that some linear codes can achieve the capacity of a BSC [10]. We assume that $\begin{bmatrix}\mathbf{G}_{11}^T & \mathbf{G}_{12}^T\end{bmatrix}^T$ is the generating matrix of such a capacity-achieving linear code. Since $\mathcal{C}_2 \subset \mathcal{C}_1$, node 1 and 2 share the transmitting subspace $\mathcal{C}_2$. The received signal at the relay node is written as

$$\mathbf{y}_R = \mathbf{x}_1 \oplus \mathbf{x}_2 \oplus \mathbf{z}_R = \mathbf{v} \oplus \mathbf{z}_R, \tag{7}$$

where $\mathbf{v} \triangleq \mathbf{x}_1 \oplus \mathbf{x}_2$. From the group property of linear codes, $\mathbf{v}$ is an element of $\mathcal{C}_1$. Thus, the relay can decode $\mathbf{v}$ instead of decoding $\mathbf{x}_1$ and $\mathbf{x}_2$ separately. If there is no decoding error, the decoded message at the relay will be given by a uniform binary vector $\mathbf{u}_R \triangleq [\mathbf{u}_{11}\ \tilde{\mathbf{u}}_2]$, where $\tilde{\mathbf{u}}_2 \triangleq \mathbf{u}_{12} \oplus \mathbf{u}_2$. The condition for the reliable uplink communication is given by

$$R_1 \le 1 - H(\epsilon_R). \tag{8}$$

In the downlink, the relay transmit $\mathbf{u}_R$ to both receiver nodes. In the binary TRC, node 1 knows $[\mathbf{u}_{11}\ \mathbf{u}_{12}]$ and node 2 knows $\mathbf{u}_2$. In this case, using the nested coding scheme [9], node 1 and 2 can reliably decode $\mathbf{u}_R = [\mathbf{u}_{11}\ \tilde{\mathbf{u}}_2]$ if

$$\begin{aligned} R_1 &\le 1 - H(\epsilon_2), \\ R_2 &\le 1 - H(\epsilon_1). \end{aligned} \tag{9}$$

After obtaining $\mathbf{u}_R$ without error, each receiver can estimate the other's message as follows:

$$\begin{aligned} &\text{Node 1: } \hat{\mathbf{u}}_2 = \tilde{\mathbf{u}}_2 \oplus \mathbf{u}_{12}, \\ &\text{Node 2: } [\hat{\mathbf{u}}_{11}\ \hat{\mathbf{u}}_{12}] = [\mathbf{u}_{11}\ (\tilde{\mathbf{u}}_2 \oplus \mathbf{u}_2)]. \end{aligned} \tag{10}$$

Summarizing (8) and (9), we have

$$\begin{aligned} R_1 &\le \min\{1-H(\epsilon_R), 1-H(\epsilon_2)\} \\ R_2 &\le \min\{R_1, 1-H(\epsilon_1)\}. \end{aligned} \tag{11}$$

The achievable region in (11) meets the outer bound (5) when $R_1 \ge R_2$. We can do similarly when $R_1 \le R_2$, and thus (5) is the capacity region for the binary TRC.

IV. Gaussian TRC

In this time, we consider a Gaussian TRC. The received signal at the relay is given by

$$\mathbf{y}_R = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}_R, \tag{12}$$

where $\mathbf{z}_R$ is the noise vector whose components are real i.i.d. Gaussian random variables with zero mean and variance σ_R^2. There are also power constraints P_1 and P_2 on the transmitted signals, i.e., $E\left\{\|\mathbf{x}_1\|^2\right\} \le P_1$ and $E\left\{\|\mathbf{x}_2\|^2\right\} \le P_2$. Similarly, the received signal at node i, $i=1$ or 2, is given by

$$\mathbf{y}_i = \mathbf{x}_R + \mathbf{z}_i, \tag{13}$$

where $\mathbf{z}_i$ is the noise vector whose components are real i.i.d. Gaussian random variables with distribution $\mathcal{N}(0,\sigma_i^2)$, and $E\left\{\|\mathbf{x}_R\|^2\right\} \le P_R$. The outer bound of the capacity region for this Gaussian TRC is given by [1]

$$\begin{aligned} R_1 &\le \min\left\{\frac{1}{2}\log\left(1+\frac{P_1}{\sigma_R^2}\right), \frac{1}{2}\log\left(1+\frac{P_R}{\sigma_2^2}\right)\right\}, \\ R_2 &\le \min\left\{\frac{1}{2}\log\left(1+\frac{P_2}{\sigma_R^2}\right), \frac{1}{2}\log\left(1+\frac{P_R}{\sigma_1^2}\right)\right\}. \end{aligned} \tag{14}$$

In Section III, we showed that a BLC can achieve the capacity region for the binary TRC. To achieve it, the group structure of the BLC played an important role. Thus, for the Gaussian TRC, we consider a lattice code, which also has a group structure. In the following argument, we will assume that $P_1 \ge P_2$.

Let us consider an n-dim lattice pair (Λ, Λ_C), where $\Lambda \subseteq \Lambda_C$. We also assume that the second moment per dimension of the Voronoi region of Λ is P_1. Then, a nested lattice code is defined as follows:

$$\mathcal{C}_1 \triangleq \{(\Lambda_C + \mathbf{s}) \cap \mathcal{R}_V(\Lambda)\}, \tag{15}$$

where $\mathcal{R}_V(\cdot)$ denotes the Voronoi region of a lattice, and $\mathbf{s} \in \mathcal{R}_V(\Lambda_C)$ is a translation vector that will be determined later. We further assume that the fine lattice Λ_C is "good for AWGN channel coding", and the coarse lattice Λ is "good for shaping" [11], [12]. The codebook $\mathcal{C}_1$ is used for node 1.

We now determine the codebook of node 2 as a subset of $\mathcal{C}_1$. To satisfy the transmit power constraint, we remove some codewords of the largest power one by one from $\mathcal{C}_1$ until the average power of remaining codewords becomes less than or equal to P_2. We call this process *Lattice reshaping*. Thus obtained set of codewords is denoted by $\mathcal{C}_2$ and used as the codebook of node 2. From the construction process, $\mathcal{C}_2$ has the largest cardinality among all subsets of $\mathcal{C}_1$, whose average power is less than or equal to P_2. Then, letting R_1 and R_2 denote the code rate of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, we have the following lemma.

Lemma 1: Consider a scaled lattice $\gamma\Lambda$, $0 \leq \gamma \leq 1$, which has P_2 as the second moment per dimension of $\mathcal{R}_V(\gamma\Lambda)$. If $\gamma\Lambda \subseteq \Lambda_C$, there exist at least a translation vector $\mathbf{s} \in \mathcal{R}_V(\Lambda_C)$ such that

$$R_2 \geq R_1 + \frac{1}{2}\log\left(\frac{P_2}{P_1}\right).$$

Proof: Consider a codebook defined as

$$\tilde{\mathcal{C}}_2(\mathbf{s}) \triangleq \left\{(\Lambda_C + \mathbf{s}) \cap \mathcal{R}_V(\gamma\Lambda)\right\},$$

and let $\tilde{P}_2(\mathbf{s})$ denote its average power. Since we have assumed $\gamma \leq 1$, $\tilde{\mathcal{C}}_2(\mathbf{s})$ is a subset of $\mathcal{C}_1$. If we let $\mathcal{V}$ be the volume of $\mathcal{R}_V(\Lambda)$, the volume of $\mathcal{R}_V(\gamma\Lambda)$ is $\gamma^n\mathcal{V}$. Since the second moment per dimension of $\mathcal{R}_V(\Lambda)$ and $\mathcal{R}_V(\gamma\Lambda)$ is P_1 and P_2 respectively, we have $\gamma = \sqrt{P_2/P_1}$. And also, since $\gamma\Lambda \subseteq \Lambda_C$,

$$|\tilde{\mathcal{C}}_2(\mathbf{s})| = \frac{\gamma^n\mathcal{V}}{\mathcal{V}_C} = 2^{nR_1}\gamma^n = 2^{nR_1}\left(\frac{P_2}{P_1}\right)^{n/2}. \tag{16}$$

Assuming that $\mathbf{s}$ is an instance of a random vector $\mathbf{S} \sim \text{Unif}\left(\mathcal{R}_V(\Lambda_C)\right)$, we have $E_{\mathbf{S}}\{\tilde{P}_2(\mathbf{S})\} = P_2$, which implies that there must be at least one $\mathbf{s} \in \mathcal{R}_V(\Lambda_C)$ such that $\tilde{P}_2(\mathbf{s}) \leq P_2$ [13]. By choosing such a $\mathbf{s}$, from (16), we have

$$R_2 = \frac{1}{n}\log|\mathcal{C}_2| \geq \frac{1}{n}\log|\tilde{\mathcal{C}}_2| = R_1 + \log\left(\frac{P_2}{P_1}\right), \tag{17}$$

which proves the lemma. □

One of the exemplary cases that lemma 1 holds is when $P_1 = P_2$ and, in this case, $\gamma = 1$. From now on, we only consider the cases that Lemma 1 holds.

Nodes 1 and 2 map their messages w_1 and w_2 to codewords $\mathbf{v}_1 \in \mathcal{C}_1$ and $\mathbf{v}_2 \in \mathcal{C}_2$, respectively. The transmitted signals of node 1 and 2 are denoted by $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, which are formed follows:

$$\begin{aligned} \mathbf{x}_1 &\triangleq (\mathbf{v}_1 + \mathbf{u}) \mod \Lambda, \\ \mathbf{x}_2 &\triangleq \mathbf{v}_2, \end{aligned} \tag{18}$$

where $\mathbf{u}$ is a random dither vector which is uniform over $\mathcal{R}_V(\Lambda)$ and independent of $\mathbf{x}_1$ and $\mathbf{x}_2$. $\mathbf{u}$ is also assumed to be known to both transmitter and receiver. Then, from the *crypto lemma* [11], $\mathbf{x}_1$ is independent of $\mathbf{v}_1$ and uniform over $\mathcal{R}_V(\Lambda)$.

Now we consider the group property of the lattice code. Let us define a codebook

$$\mathcal{C} \triangleq \{\Lambda_C \cap \mathcal{R}(\Lambda)\} = \{\Lambda_C \mod \Lambda\}. \tag{19}$$

If we define $\tilde{\mathbf{v}}_1 = (\mathbf{v}_1 - \mathbf{s}) \mod \Lambda$ and $\tilde{\mathbf{v}}_2 = (\mathbf{v}_2 - \mathbf{s}) \mod \Lambda$, then $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$ are elements of $\mathcal{C}$, and it can be easily seen that $\mathbf{v} \triangleq (\tilde{\mathbf{v}}_1 + \tilde{\mathbf{v}}_2) \mod \Lambda$ is also an element of $\mathcal{C}$. This property is analogous to the group property of BLC. From the *inflated lattice lemma* [14], the received signal at the relay (12) is transformed to the following modulo lattice additive (MLAN) channel signal:

$$\begin{aligned} \tilde{\mathbf{y}}_R &= (\alpha\mathbf{y}_R - \mathbf{u} - 2\mathbf{s}) \mod \Lambda \\ &= (\mathbf{v} + \tilde{\mathbf{z}}_R) \mod \Lambda, \end{aligned} \tag{20}$$

where α is a scaling factor that will be determined later, and

$$\tilde{\mathbf{z}}_R \triangleq -(1-\alpha)(\mathbf{x}_1 + \mathbf{x}_2) + \alpha\mathbf{z}_R. \tag{21}$$

As in the binary case in Section III, the relay can decode $\mathbf{v}$ instead of decoding $\mathbf{v}_1$ and $\mathbf{v}_2$ separately. If $\mathbf{v}_1$ is chosen from $\mathcal{C}_1$ equally likely, from the *crypto lemma* [11], $\mathbf{v}$ is independent of $\mathbf{v}_2$ and $\mathbf{x}_2$, and uniform over $\mathcal{C}$. Then, since $\mathbf{v}$ is independent of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{z}_R$, it is also independent of the effective noise $\tilde{\mathbf{z}}_R$. Thus, the capacity of this MLAN channel is given by

$$\begin{aligned} C_1 &= \frac{1}{n}I\left(\mathbf{V};\tilde{\mathbf{Y}}_R\right) \\ &= \frac{1}{n}h\left(\tilde{\mathbf{Y}}_R\right) - \frac{1}{n}h\left(\tilde{\mathbf{Z}}_R \mod \Lambda\right) \\ &\geq \frac{1}{2}\log\left(\frac{P_1}{G(\Lambda)}\right) - \frac{1}{n}h\left(\tilde{\mathbf{Z}}_R\right), \end{aligned} \tag{22}$$

where $G(\Lambda)$ is the normalized second moment [11] of Λ. From the mutual independence of $\tilde{\mathbf{X}}_1$, $\mathbf{X}_2$, and $\mathbf{Z}_R$, we have

$$\frac{1}{n}E\left\{\left\|\tilde{\mathbf{Z}}_R\right\|^2\right\} \leq (1-\alpha)^2(P_1+P_2) + \alpha^2\sigma_R^2. \tag{23}$$

If we choose the value of α to minimize the right hand side of (23),

$$\frac{1}{n}E\left\{\left\|\tilde{\mathbf{Z}}_R\right\|^2\right\} \leq \frac{\sigma_R^2(P_1+P_2)}{P_1+P_2+\sigma_R^2}. \tag{24}$$

Then, since a Gaussian random vector has the largest entropy for a given second moment, we have

$$\frac{1}{n}h\left(\tilde{\mathbf{Z}}_R\right) \leq \frac{1}{2}\log\left(2\pi e\frac{\sigma_R^2(P_1+P_2)}{P_1+P_2+\sigma_R^2}\right), \tag{25}$$

and, thus, using (25) in (22),

$$C_1 \geq \frac{1}{2}\log\left(\frac{P_1}{P_1+P_2} + \frac{P_1}{\sigma_R^2}\right) - \frac{1}{2}\log\left(2\pi eG(\Lambda)\right). \tag{26}$$

Since we have assumed that Λ is good for shaping [11], $\log(2\pi eG(\Lambda))$ can be made arbitrarily small. For sufficiently large dimension n, the existence of nested lattice pairs (Λ, Λ_C), where Λ is good for shaping and, at the same time, Λ_C is good for AWGN channel coding, is shown in [12]. Thus, by assuming the use of such lattice pairs, the uplink achievable region is given by

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log\left(\frac{P_1}{P_1+P_2} + \frac{P_1}{\sigma_R^2}\right) + o(1), \\ R_2 &\leq \frac{1}{2}\log\left(\frac{P_2}{P_1+P_2} + \frac{P_2}{\sigma_R^2}\right) + o(1), \end{aligned} \tag{27}$$

where $\lim_{n\to\infty} o(1) = 0$.

For downlink communication, we can rely on the nested coding scheme [9]. The relay transmits $\mathbf{v}$ to both receiver nodes using a Gaussian random codebook. The receivers do the jointly-typical decoding. The size of this codebook is 2^{nR_1}. However, since $\mathbf{v} = (\tilde{\mathbf{v}}_1 + \tilde{\mathbf{v}}_2) \mod \Lambda$ and node 1 already knows $\tilde{\mathbf{v}}_1$, node 1 only needs to search for a jointly typical sequence in a subset of the codebook, whose size is 2^{nR_2}. Thus, the downlink achievable region is given by

$$R_1 \le \frac{1}{2}\log\left(1 + \frac{P_R}{\sigma_2^2}\right), \\ R_2 \le \frac{1}{2}\log\left(1 + \frac{P_R}{\sigma_1^2}\right). \quad (28)$$

After obtaining $\mathbf{v}$ without error, each receiver can estimate the other's message as follows:

$$\text{Node 1: } \hat{\mathbf{v}}_2 = (\mathbf{v} - \tilde{\mathbf{v}}_1 + \mathbf{s}) \mod \Lambda, \\ \text{Node 2: } \hat{\mathbf{v}}_1 = (\mathbf{v} - \tilde{\mathbf{v}}_2 + \mathbf{s}) \mod \Lambda. \quad (29)$$

Summarizing (27) and (28), when $n \to \infty$, we have

$$R_1 \le \min\left\{\frac{1}{2}\log\left(\frac{P_1}{P_1+P_2} + \frac{P_1}{\sigma_R^2}\right), \frac{1}{2}\log\left(1 + \frac{P_R}{\sigma_2^2}\right)\right\}, \\ R_2 \le \min\left\{\frac{1}{2}\log\left(\frac{P_2}{P_1+P_2} + \frac{P_2}{\sigma_R^2}\right), \frac{1}{2}\log\left(1 + \frac{P_R}{\sigma_1^2}\right)\right\}. \quad (30)$$

Note that, for the symmetric case, (30) coincides with the result of [6]. Comparing (30) with the upper bound (14), we can easily verify that the rate gap is always less than or equal to $1/2$ bit for any choices of channel parameters, such as the transmit power and noise variance. Especially, as the signal to noise ratio increases, this gap vanishes and the lower bound approaches asymptotically the upper bound.

The achievable rate region (30) is shown in Fig. 2. In Fig. 2, it is assumed that $\sigma_1^2 = \sigma_2^2 = \sigma_R^2 = \sigma^2$ and $P_1 = P_2 = P \ll P_R$. Thus, in this case, the uplink is the limiting factor. For comparison, cut-set bound (1) and DF region (2) are also shown in Fig. 2. Clearly, we can think of the time sharing between DF relaying and lattice scheme, and thus the convex hull of the two achievable regions, which is depicted as the shaded region in Fig. 2, is achievable.

V. Conclusion

We studied achievable rate regions for TRC. First, we have seen that the subspace-sharing of linear codes can achieve the capacity region for a binary TRC. Then, we proposed the subset-sharing of lattice codes for a Gaussian TRC. It was also shown that, in some cases, the proposed lattice-coding scheme achieves to within $1/2$-bit capacity region of the Gaussian TRC. At high SNR regimes, the achievable region asymptotically approaches the capacity upper bound.

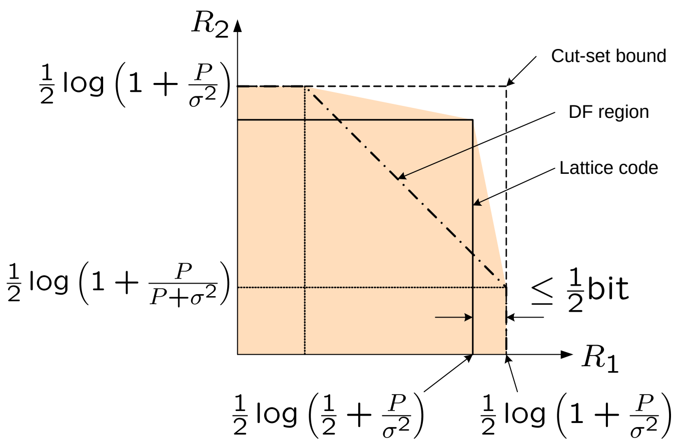

Fig. 2. Achievable region of the Gaussian TRC

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