

Fig. 5. EKF frequency estimate when $Q^d > Q^a$, $\epsilon^d < \epsilon^a$, and $R^d > R^a$. The target signal is given by the solid line.

TABLE I
MEAN AND STANDARD DEVIATION OF FREQUENCY ESTIMATE ERROR

SNR	ANF - "best"		EKF - "best"		EKF - "actual"	
	mean(MSE)	std(MSE)	mean(MSE)	std(MSE)	mean(MSE)	std(MSE)
17dB	0.4185	0.0342	0.3252	0.0255	2.9740	0.0329
13dB	0.4497	0.0534	0.4493	0.0290	3.9001	0.0765
10dB	0.5287	0.0506	0.4904	0.0374	4.7783	0.1572
7dB	0.6361	0.0908	0.7268	0.0517	25.782	17.868

design tradeoffs between balancing noise rejection with tracking at a maximal slew rate. The nonlinearity inherent in this problem makes this tradeoff nonobvious, critical to the performance of the filter, and counter to the usual optimality guidelines for Kalman filtering. The performance penalties for overestimation and underestimation of the noise covariances are further illustrated via simulation results. These demonstrate the importance of designing a sufficiently conservative filter and might explain why so few successful applications of extended Kalman filters have been reported.

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Analysis and Optimization of Subset Averaged Median Filters

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Abstract—By analyzing the subset averaged median (SAM) filter based on threshold decomposition, we show that the class of SAM filters is identical to the class of extended threshold Boolean filters (ETBF's) with the *extended self-dual* property. This result indicates that the class of SAM filters encompasses a variety of digital filters such as linear finite impulse response (FIR), weighted median, symmetric L-filters, and any filter defined by a linear combination of these filters. A procedure for determining an optimum SAM filter in the mean square error (MSE) sense is developed. It is shown that the optimization of SAM filters may result in a FIR Wiener filter when the input is Gaussian and in a median-type filter for non-Gaussian inputs.

I. INTRODUCTION

The subset averaged median (SAM) filter is a multistage digital filter [1], [2] containing median subfilters. In this filter, the final output is a weighted average calculated over the outputs of median subfilters. SAM filters¹ were introduced in [3] and [4] as an extension of median filters.

In this correspondence, we shall show that the class of ETBF's [5], [6] with an *extended self-dual* property is identical to the SAM class filter. This result indicates that the class of SAM filters encompasses a variety of filters such as linear FIR, weighted median [7], symmetric L-filters [8], and any filter defined by a linear combination of these filters. In addition, we shall develop a procedure for determining the best SAM filter in the mean square error (MSE) sense. It will be observed through computer simulation that the optimization of SAM filters may result in an optimal FIR Wiener filter when the input is zero-mean Gaussian and in a nonlinear filter that outperforms the Wiener filter for non-Gaussian inputs.

II. THE SAM FILTER AND THE ETBF

In this section, we review the definitions of SAM filters and ETBF's, and introduce some subclasses of these filters such as *full*-SAM filters and *extended self-dual* ETBF's.

A. The SAM Filter

Consider a nonrecursive filter that evaluates its output from the input sequence $X(n) = (X_1(n), X_2(n), \dots, X_N(n))$ taken from the window at time n where $X_j(n)$ is the j th input sample from the left of the window and N is the window size. We define the i th median subfilter denoted by $F_i(X)$ as follows:

$$F_i(X) = \text{med}(X_i) \quad (1)$$

where X_i is a subsequence of X and $F_i(X)$ takes the median value of the inputs in X_i . Here and in the rest of this correspondence, the time index n is dropped from $X(n)$ and $X_j(n)$ to simplify the notation. It is assumed that the number of inputs in X_i is an odd

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¹SAM filters in [3] calculate the simple average of median subfilter outputs.

integer and that $F_i(\mathbf{X})$ is an identity operator when X_i consists of only one element. The SAM filter is defined as

$$\text{SAM}(\mathbf{X}) = \sum_{i=1}^P a_i F_i(\mathbf{X}) \quad (2)$$

where $\{a_i\}$ are constants and P is the number of median subfilters, $1 \leq P \leq 2^{N-1}$. This filter reduces to a linear FIR filter when every X_i has only one element. When $P = 2^{N-1}$, the SAM filter evaluates all possible median subfilters. We call such a SAM filter a full-SAM (F-SAM) filter. For example, when $N = 4$

$$\begin{aligned} \text{F-SAM}(\mathbf{X}) = & \sum_{i=1}^4 a_i X_i \\ & + a_5 \text{med}(X_1, X_2, X_3) \\ & + a_6 \text{med}(X_1, X_2, X_4) \\ & + a_7 \text{med}(X_1, X_3, X_4) \\ & + a_8 \text{med}(X_2, X_3, X_4). \end{aligned} \quad (3)$$

B. The ETBF

The ETBF is defined based on threshold decomposition architecture [9]. Assume that each input sample X_j is $(Q+1)$ -valued, i.e., $X_j \in \{0, 1, \dots, Q\}$. The output of an ETBF is given by

$$\text{ETBF}(\mathbf{X}) = \sum_{q=1}^Q f(T_q(\mathbf{X})) \quad (4)$$

where $T_q(\mathbf{X}) = (T_q(X_1), T_q(X_2), \dots, T_q(X_N))$ is the threshold operator defined as $T_q(X_j) = 1$ if $X_j \geq q$ and 0 otherwise, and $f(\cdot)$ is a function that associates to each binary vector \mathbf{X} the real number $f(\mathbf{X})$. The ETBF becomes a threshold Boolean [5] (stack [9]) filter when $f(\cdot)$ is a Boolean (positive Boolean) function, and a linear FIR filter when $f(\mathbf{X})$ is the weighted average calculated over the elements of \mathbf{X} . If $f(\mathbf{X})$ takes a linear combination of the ordered elements of \mathbf{X} , then the corresponding ETBF is an L-filter. Alternative expressions that directly specify the ETBF on multilevels without using threshold decomposition architecture and procedures for designing ETBF's have been developed in [6] and [10].

Note that an ETBF in (4) can be completely specified by the function $f(\cdot)$. This is called the extended Boolean function (EBF), and the table listing all possible binary input sequences of length N and the corresponding output values is called the extended truth table. In defining ETBF's, we shall exclude those constant Boolean functions $f(\mathbf{X}^r) = 1 \forall r$ and $f(\mathbf{X}^r) = 0 \forall r$. Define for $r = 1, 2, \dots, 2^N - 1$

$$l_r = f(\mathbf{X}^r) - f(\mathbf{X}^{r-1}) \quad (5)$$

where \mathbf{X}^r is a binary vector that is the radix 2 representation of r , and $f(\mathbf{X}^0) = 0$.

Definition: An EBF $f(\cdot)$ is extended self-dual if l_r is symmetric, i.e., $l_r = l_{2^N-r}$, $r = 1, 2, \dots, 2^N - 1$.

An ETBF with extended self-dual $f(\cdot)$ is said to have an extended self-dual property. When $f(\cdot)$ is a Boolean function, the symmetry in l_r implies the self-duality $f(\bar{\mathbf{X}}) = f(\mathbf{X})$ and vice versa. This is shown in the following property.

Property 1: Assume that a Boolean function $f(\cdot)$ is nonconstant, i.e., neither $f(\mathbf{X}) = 0 \forall \mathbf{X}$ nor $f(\mathbf{X}) = 1 \forall \mathbf{X}$ hold. Then $f(\cdot)$ is self-dual if and only if l_r is symmetric.

Proof: Necessity—Consider $f(\mathbf{X}^r)$ and $f(\mathbf{X}^{r-1})$. If $f(\mathbf{X})$ is self-dual, then $f(\bar{\mathbf{X}}^r) = f(\mathbf{X}^{2^N-r-1})$ and $f(\mathbf{X}^r) + f(\mathbf{X}^{2^N-r-1}) = 1$. Similarly, $f(\mathbf{X}^{r-1}) + f(\mathbf{X}^{2^N-r}) = 1$. Thus, $l_r = f(\mathbf{X}^r) - f(\mathbf{X}^{r-1}) = f(\mathbf{X}^{2^N-r}) - f(\mathbf{X}^{2^N-r-1}) = l_{2^N-r}$.

Sufficiency—Since l_r is symmetric, $f(\mathbf{X}^r) + f(\mathbf{X}^{2^N-r-1}) = f(\mathbf{X}^{2^N-r}) + f(\mathbf{X}^{r-1}) \forall r$ and $f(\mathbf{X}^1) + f(\mathbf{X}^{2^N-2}) = f(\mathbf{X}^2) + f(\mathbf{X}^{2^N-3}) = \dots = f(\mathbf{X}^{2^{N-1}-1}) + f(\mathbf{X}^{2^N-1})$. This equality implies self-duality when $f(\mathbf{X}^r) + f(\mathbf{X}^{2^N-r-1}) = 1 \forall r$. When $f(\mathbf{X}^r) + f(\mathbf{X}^{2^N-r-1}) = 0 \forall r$, either $f(\mathbf{X}^r) = 0 \forall r$ or $f(\mathbf{X}^r) = 1 \forall r$ should hold. This contradicts the assumption. ■

Median and weighted median filters are specified by self-dual Boolean functions and, thus, they are subclasses of extended self-dual ETBF's. Next, we show that a linear FIR filter is also extended self-dual.

Property 2: A linear FIR filter is an extended self-dual ETBF.

Proof: Define the extended Boolean function of an FIR filter as $f(\mathbf{X}^r) = \sum_{j=1}^N h_j x_j^r$ where $\{h_j\}$ is the impulsive response and x_j^r is the j th entry of \mathbf{X}^r . Then $l_r = \sum_{j=1}^N h_j (x_j^r - x_j^{r-1}) = \sum_{j=1}^N h_j (x_j^{2^N-r} - x_j^{2^N-r-1}) = l_{2^N-r}$. The second equality comes from $x_j^r + x_j^{2^N-r-1} = 1$. ■

In a similar manner, it can be seen that an L-filter is extended self-dual if its linear combination vector is symmetric. Now consider a multistage filter possessing extended self-dual subfilters and determining its final output by calculating a weighted average of the subfilter outputs. This kind of a multistage filter is also extended self-dual, as shown below.

Property 3: A linear combination of extended self-dual ETBF's is also extended self-dual.

Proof: Define a linear combination of extended self-dual filters as $f(\mathbf{X}) = \sum_{k=1}^K c_k f_k(\mathbf{X})$ where $f_k(\mathbf{X})$ is an extended self-dual EBF and K is the number of such EBF's. Since $l_r = \sum_{k=1}^K c_k (f_k(\mathbf{X}^r) - f_k(\mathbf{X}^{r-1})) = \sum_{k=1}^K c_k (f_k(\mathbf{X}^{2^N-r}) - f_k(\mathbf{X}^{2^N-r-1})) = l_{2^N-r}$, $f(\mathbf{X})$ is extended self-dual. ■

This property indicates that any filter defined by a linear combination of linear FIR, weighted median and symmetric L-filters is an extended self-dual ETBF.

III. PROPERTIES OF SAM FILTERS

In this section, it is shown that the class of SAM filters is identical to the class of extended self-dual ETBF's.

Property 4: Any SAM filter can be represented as an extended self-dual ETBF.

Proof: Consider $\text{SAM}(\mathbf{X}) = \sum_{i=1}^P a_i F_i(\mathbf{X})$. Since the median operator $F_i(\cdot)$ has the threshold decomposition property [11] and commutes with every nondecreasing function [12], then $\text{SAM}(\mathbf{X}) = \sum_{i=1}^P a_i \sum_{q=1}^Q T_q(F_i(\mathbf{X})) = \sum_{q=1}^Q \sum_{i=1}^P a_i F_i(T_q(\mathbf{X}))$. Now, from Property 3, the self-duality of $\text{SAM}(\mathbf{X})$ follows. ■

The converse of Property 4 can be proven by exploiting the fact that the logical exclusive-OR operator can be represented as a F-SAM filter. In the following lemma, we consider a F-SAM filter whose input is given by a subsequence \mathbf{X}_s of a binary sequence $\mathbf{X} = (X_1, X_2, \dots, X_N)$. The length of \mathbf{X}_s , denoted by M , is assumed to be an odd integer $3 \leq M \leq N$. It is also assumed that the median subfilters of F-SAM(\mathbf{X}_s) are arranged so that the i th subfilter $F_i(\mathbf{X}_s)$ has a window of size $2k-1$ if $g(k-1) + 1 \leq i \leq g(k)$ where $g(0) = 0$, $g(k) = \sum_{j=1}^k \binom{M}{2j-1}$, and $1 \leq k \leq \frac{(M+1)}{2}$.

Lemma 1: The logical exclusive-OR operator, denoted by $\text{xor}(\cdot)$, can be represented as

$$\text{xor}(\mathbf{X}_s) = \sum_{k=1}^{\frac{M+1}{2}} c_k \sum_{i=g(k-1)+1}^{g(k)} F_i(\mathbf{X}_s) \quad (6)$$

which is a F-SAM filter with $a_i = c_k$ whenever $g(k-1) + 1 \leq i \leq g(k)$. Here, $\{c_k\}$ are constants that can be obtained by solving (8) presented below.

Proof: The output of $xor(\cdot)$ is one if the number of one's in X_s , say m , is odd; otherwise it is zero. Let β_k^m be the value of $\sum_{i=g(k-1)+1}^{g(k)} F_i(X_s)$ for an input X_s with m one's. Note that β_k^m is the number of subfilters of size $2k-1$ producing one for an input with m one's. Of course, $\beta_1^m = m$, the size $2k-1 \leq M$, and $m \leq M$ where M is the length of X_s . Since median subfilters of size $2k-1$ always produce zero when $k > m$, then $\beta_k^m = 0$ if $k > m$; otherwise $\beta_k^m > 0$. Using these relations, (6) can be rewritten as

$$c_1\beta_1^m + c_2\beta_2^m + \dots + c_{\frac{M+1}{2}}\beta_{\frac{M+1}{2}}^m = \begin{cases} 1 & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even} \end{cases} \quad (7)$$

where $m = 1, \dots, M$ and $\beta_k^m = 0$ if $k > m$, or in matrix-vector form as follows:

$$\beta c = b \quad (8)$$

where

$$\beta = \begin{bmatrix} \beta_1^1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \beta_1^{\frac{M-1}{2}} & \dots & \beta_{\frac{M-1}{2}}^{\frac{M-1}{2}} & 0 \\ \beta_1^{\frac{M+1}{2}} & \dots & \dots & \beta_{\frac{M+1}{2}}^{\frac{M+1}{2}} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_1^M & \beta_2^M & \dots & \beta_{\frac{M+1}{2}}^M \end{bmatrix}, c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{\frac{M-1}{2}} \\ c_{\frac{M+1}{2}} \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Now it is sufficient to show that the rank of β is $\frac{M+1}{2}$. Consider median subfilters of size $2k-1$ with X_s having m one's. Note that the total number of such median subfilters is $\beta_k^M = M C_{2k-1}$ and that β_k^{M-m} is the number of median subfilters of size $2k-1$ producing one for the input \bar{X}_s where \bar{X}_s is the logical complement of X_s . Thus, β_k^{M-m} is equal to the number of median subfilters producing zero for the input X_s having m one's, and we get $\beta_k^m = \beta_k^M - \beta_k^{M-m}$ for $k = 1, 2, \dots, \frac{M+1}{2}$ and $m = 1, 2, \dots, M-1$. To evaluate the rank of β , we proceed by elementary row operations. Denote by R_m , $m = 1, 2, \dots, M$, the m th row of β . For each m , $1 \leq m \leq \frac{M-1}{2}$ apply the row operations $R_{M-m} \rightarrow R_{M-m} - R_m$ and then the operation $R_{M-m} \rightarrow R_{M-m} - R_m$. The first $\frac{(M-1)}{2}$ rows and the M th row of the resulting matrix, say β' , are identical to those of β , and the other rows of β' are $\mathbf{0}$ vectors. The rank of β' is $\frac{(M+1)}{2}$, since $\beta_k^m > 0$ for $k \leq m$. This completes the proof. ■

For example, if $X_s = (x_1, x_2, x_4)$ then F-SAM(X_s) may be expressed as F-SAM(X_s) = $a_1x_1 + a_2x_2 + a_3x_4 + a_4med(X_s)$. Now (8) is written as

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (9)$$

and we get $c_1 = 1$, $c_2 = -2$. Thus, $xor(X_s) = x_1 + x_2 + x_4 - 2med(x_1, x_2, x_4)$.

Property 5: Any extended self-dual ETBF can be represented as a SAM filter.

Proof: Consider the SAM filter in (2) and the ETBF in (4). The existence of a F-SAM filter equivalent to a self-dual ETBF can be shown by finding the relationships between $\{a_i\}$ and l satisfying $\sum_{i=1}^P a_i F_i(X^r) = f(X^r)$ or equivalently $\sum_{i=1}^P a_i \{F_i(X^r) - F_i(X^{r-1})\} = l_r$ for all $r = 1, 2, \dots, R$ where $P = 2^{N-1}$, $R = 2^N - 1$ and $F_i(X^0) = 0$. This relation can be written as

$$B \cdot a = l \quad (10)$$

where

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1P} \\ b_{21} & b_{22} & \dots & b_{2P} \\ \vdots & \vdots & \ddots & \vdots \\ b_{R1} & b_{R2} & \dots & b_{RP} \end{bmatrix}, a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_P \end{bmatrix}, l = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_R \end{bmatrix}$$

TABLE I
PARAMETERS OF THE F-SAM IN (3) CORRESPONDING
TO AN EXTENDED SELF-DUAL ETBF WITH $N = 4$

parameters of SAM	$l_i = l_{2N-i}$
a_1	$l_1 + l_2 + l_3 + l_4 + l_5 + l_6 + l_7 + l_8$
a_2	$l_1 + l_2 + l_3 + l_4$
a_3	$l_1 + l_2$
a_4	l_1
a_5	$-\frac{1}{2}l_1 - l_2 - \frac{1}{2}l_3 + \frac{1}{2}l_5 + l_6 + \frac{1}{2}l_7$
a_6	$-\frac{1}{2}l_1 - \frac{1}{2}l_3 + \frac{1}{2}l_5 + \frac{1}{2}l_7$
a_7	$-\frac{1}{2}l_1 + \frac{1}{2}l_3 - \frac{1}{2}l_5 + \frac{1}{2}l_7$
a_8	$-\frac{1}{2}l_1 + \frac{1}{2}l_3 + \frac{1}{2}l_5 - \frac{1}{2}l_7$

and $b_{ri} = F_i(X^r) - F_i(X^{r-1})$. Now it is sufficient to show that the rank of B is P . By applying elementary row operations $b_r \rightarrow b_{r-1} + b_r$ to each row b_r of B with the exception of the first row, we get

$$B' = \begin{bmatrix} F_1(X^1) & F_2(X^1) & \dots & F_P(X^1) \\ F_1(X^2) & F_2(X^2) & \dots & F_P(X^2) \\ \vdots & \vdots & \ddots & \vdots \\ F_1(X^R) & F_2(X^R) & \dots & F_P(X^R) \end{bmatrix} \quad (11)$$

We proceed with elementary column operations. Denote by $F_i(X^r) = med(X_i^r)$ where X_i^r is a subsequence of X^r . Consider $xor(X_i^r)$ in (6). Successive use of elementary column operations based on the result in Lemma 1 yields

$$B'' = \begin{bmatrix} xor(X_1^1) & xor(X_2^1) & \dots & xor(X_P^1) \\ xor(X_1^2) & xor(X_2^2) & \dots & xor(X_P^2) \\ \vdots & \vdots & \ddots & \vdots \\ xor(X_1^R) & xor(X_2^R) & \dots & xor(X_P^R) \end{bmatrix} \quad (12)$$

where for $i = 1, 2, \dots, N$, $xor(X_i^r) = F(X_i^r) = x_i$. For example, if $X_i^r = (x_1^r, x_2^r, x_4^r)$ then the elements of the i th column of B' are replaced with $\sum_{i=1}^3 F_i(X_i^r) - 2F_i(X_i^r) = x_1^r + x_2^r + x_4^r - 2med(X_i^r) = xor(X_i^r)$, $1 \leq r \leq P$ (see (9)). The rank of B'' is P if the $P \times P$ matrix $(B'')^T B''$ is nonsingular where T means transpose. It is not difficult to derive $\sum_{r=1}^R [xor(X_i^r)]^2 = 2^{N-1}$ and $\sum_{r=1}^R xor(X_i^r) xor(X_j^r) = 2^{N-2}$ for any i and j . From these results

$$(B'')^T B'' = 2^{N-2} \begin{bmatrix} 2 & 1 & \dots & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \dots & 1 & 2 \end{bmatrix} \quad (13)$$

Again using elementary row operations, $(B'')^T B''$ can be converted into an upper triangular matrix with nonzero diagonal entries. This completes the proof. ■

A F-SAM filter identical to a given extended self-dual ETBF can be obtained by solving (10). For the F-SAM filter in (3), (10) is solved and the results are summarized in Table I. From these results, we can see the relationship between the SAM filter and the weighted median filter with $N = 4$. For example, if a weighted median filter calculates the median of $(X_1, X_1, X_2, X_2, X_2, X_3, X_3, X_4, X_4)$ then the corresponding F-SAM filter is given by F-SAM(X) = $1/2\{med(X_1, X_2, X_3) + med(X_1, X_2, X_4) - med(X_1, X_3, X_4) + med(X_2, X_3, X_4)\}$.

IV. OPTIMIZATION OF SAM FILTERS

Suppose that the observation signal $X(n)$ is a distorted version of a desired signal $S(n)$, which is a wide-sense stationary random signal. Our goal is to estimate $S(n)$ from a set $\{X(n), X(n-1), \dots, X(n-N+1)\}$. These N samples will be denoted by \mathbf{X} as before, and the time index n will be dropped from $S(n)$ for the sake of simplicity in notation.

Consider the MSE

$$J(\mathbf{a}) = E[(S - \sum_{i=1}^P a_i F_i(\mathbf{X}))^2] \quad (14)$$

where $\mathbf{a} = [a_1, \dots, a_P]^T$ and $E[\cdot]$ is the expectation operator. The minimization of $J(\mathbf{a})$ with respect to \mathbf{a} is a standard linear estimation problem whose solution is given by a vector \mathbf{a}_{opt} satisfying

$$\Psi \mathbf{a}_{opt} = \Phi \quad (15)$$

where $\Psi = E[\mathbf{F}(\mathbf{X}) \mathbf{F}^T(\mathbf{X})]$ is the $P \times P$ correlation matrix with $\mathbf{F}(\mathbf{X}) = [F_1(\mathbf{X}), \dots, F_P(\mathbf{X})]^T$, and $\Phi = E[S\mathbf{F}(\mathbf{X})]$ is the P dimensional cross-correlation vector. It is simple to design an optimal SAM filter using (15) when Ψ and Φ are known and the number of subfilters P is small. In practice, however, Ψ and Φ should be estimated from observation; P is often large for some SAM filters such as F-SAM filters with $P = 2^{N-1}$. These facts will limit the use of this method.

In the following, we present a design example in which optimal SAM filters are designed for an autoregressive (AR) signal contaminated by additive white noise. Consider the first order AR signal

$$S(n) = \rho S(n-1) + W(n) \quad (16)$$

where $W(n)$ is a white Gaussian noise with zero-mean and variance $\sigma_W^2 = 1$ and ρ is a constant set at 0.9. The observed sequence $\{X(n)\}$ is given by

$$X(n) = S(n) + \eta(n) \quad (17)$$

where $\{\eta(n)\}$ are zero-mean independent, identically distributed (i.i.d.) random variables having ϵ -contaminated Gaussian distribution [12]. The distribution function of this noise is given by

$$P_\eta(x) = (1 - \epsilon)\Theta(x/\sigma_1) + \epsilon\Theta(x/\sigma_2) \quad (18)$$

where $\Theta(\cdot)$ is a normalized Gaussian distribution function, σ_1^2 is the variance of the nominal distribution, and σ_2^2 is the variance of the outlier ($\sigma_2 \gg \sigma_1$). In our simulations, we have selected $\epsilon = 0$ (Gaussian noise) and $\epsilon = 0.1$ (heavy tailed noise). We design two types of SAM filters of span four: one is the causal F-SAM filter defined in (3); the other is a SAM filter, which we call a successive-SAM (S-SAM) filter defined as

$$\begin{aligned} \text{S-SAM}(\mathbf{X}) = & \sum_{i=1}^4 a_i X_i \\ & + a_5 \text{med}(X_1, X_2, X_3) \\ & + a_6 \text{med}(X_2, X_3, X_4). \end{aligned} \quad (19)$$

Note that this filter considers only those subfilters with successive input data. The statistics required for designing the filters are estimated from computer-generated data consisting of 500 000 samples where it is assumed we have access to both $\{S(n)\}$ and $\{X(n)\}$: at each point, $\mathbf{F}(\mathbf{X})\mathbf{F}^T(\mathbf{X})$ and $S\mathbf{F}(\mathbf{X})$ are evaluated and the average of $\mathbf{F}(\mathbf{X})\mathbf{F}^T(\mathbf{X})$ and the average of $S\mathbf{F}(\mathbf{X})$ are used as estimates of Ψ and Φ , respectively. For comparison, optimal linear FIR (Wiener) filters are also designed. Wiener filters are derived both theoretically and experimentally in the case of Gaussian noise. The results are

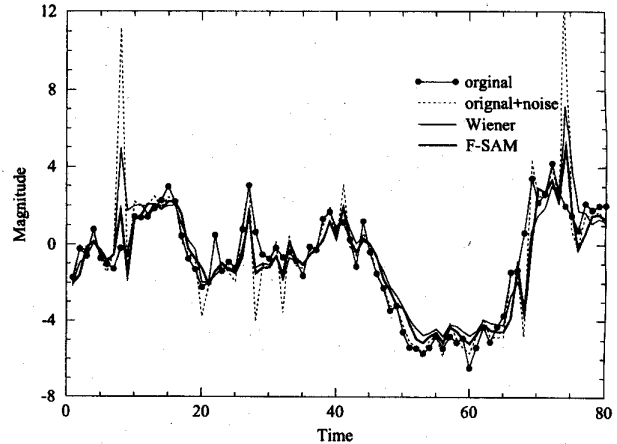


Fig. 1. A portion of the original AR, noisy, and filtered signals for ϵ -contaminated Gaussian noise ($\epsilon = 0.1$, $\sigma_1^2 = 0.2$, and $\sigma_2^2 = 20$).

TABLE II
OPTIMAL SAM, MEDIAN, AND FIR WIENER FILTERS OF SPAN FOUR WHERE \mathbf{h} IS THE IMPULSIVE RESPONSE OF THE WIENER FILTER: (a) GAUSSIAN NOISE WITH $\sigma_1^2 = 0.25$; (b) ϵ -CONTAMINATED GAUSSIAN NOISE WITH $\epsilon = 0.1$, $\sigma_1^2 = 0.2$, $\sigma_2^2 = 20$

Wiener(theoretical)	\mathbf{h}	(0.8235, 0.1308, 0.0208, 0.0038)
	MSE	0.2059
Wiener(empirical)	\mathbf{h}	(0.8237, 0.1306, 0.0205, 0.0041)
	MSE	0.2068
F-SAM	\mathbf{a}	(0.8228, 0.1287, 0.0197, 0.004, 0.0035, 0.0007, -0.0009, 0.0005)
	MSE	0.2068
S-SAM	\mathbf{a}	(0.8227, 0.1290, 0.0194, 0.004, 0.0037, 0.0001)
	MSE	0.2068

(a)

Wiener(empirical)	\mathbf{h}	(0.4553, 0.2280, 0.1196, 0.0742)
	MSE	0.9955
F-SAM	\mathbf{a}	(0.1629, -0.0495, -0.0572, -0.0638, 0.5222, 0.4068, 0.1610, -0.1518)
	MSE	0.6886
S-SAM	\mathbf{a}	(0.2326, -0.0331, -0.0904, 0.0067, 0.7855, 0.0209)
	MSE	0.7367

(b)

summarized in Table II. As expected, the theoretical and empirical Wiener filters in Table II (a) are almost identical. It is interesting to note in Table II (a) that the SAM filters are essentially the same as the Wiener filter. This is because the Wiener filter is the optimal minimum MSE estimator when the noise $\eta(n)$ is Gaussian. The results in Table II (b) demonstrate that the SAM filters outperform the Wiener filter for ϵ -contaminated noise. Note that MSE values associated with the F-SAM and S-SAM filters are close to each other. In this case the S-SAM filter may be preferred, due to its simplicity in implementation, to the F-SAM filter. Fig. 1 shows a portion of the original AR, noisy and filtered signals. (Since F-SAM and S-SAM filtered signals look almost identical, we show only the former.) It is seen that the SAM filter is reasonably robust to outliers, and performs like the Wiener filter in portions of the signal where outliers are nonexistent.

V. CONCLUSIONS

It was shown that the class of F-SAM filters is equivalent to that of ETBF's with an extended self-dual property. This result indicates that F-SAM filters include linear FIR, weighted median, symmetric

L-filters, and any filter defined by a linear combination of these filters. The optimization of the SAM filter under the MSE criterion has been investigated. Computer simulations showed that the optimization of F-SAM filters may result in a FIR Wiener filter when the input is Gaussian and in a median-type filter for non-Gaussian inputs. Future work in this direction will concentrate on investigating various types of SAM filters, which include F-SAM, and S-SAM, and examining the performance and complexity of these filters.

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A Direct Solution for Blind Separation of Sources

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Abstract—This correspondence proposes a direct estimation of the mixing matrix in the source separation problem. As a new result, we prove that for non-Gaussian sources, the mixing matrix can be obtained by solving a fourth-degree polynomial equation that derives from nonlinear equations only involving fourth-order cumulants.

I. INTRODUCTION

Most of the recent solutions for blind separation of sources are based on estimation of a separation matrix, such that the product of the observation (mixture) vector multiplied by the separation matrix gives statistically independent signals. The algorithms, adaptive or not, use independence criteria based on higher order statistics. Methods include minimization of one criterion [2], [1], [6] or of one contrast function [4], [7], or cancellation of multiple criteria [5], [9].

For instantaneous mixtures of two sources, direct methods, which consist of directly estimating the mixing matrix from the mixtures, are also possible. A direct method has already been suggested by Comon [3], who used cumulants up to fourth-order (that is, including second-order) and proposed solutions obtained by rooting two second-degree polynomial equations. In this correspondence, we prove that a direct solution can be obtained by using only fourth-order cumulants if the two sources are non-Gaussian.

The mixture model, notations, and equations based on second- and fourth-order cross-cumulants, are given in Section II. In Section III, we present solutions obtained using fourth-order cumulants. A short discussion, including experimental results, constitutes Section IV.

II. MODEL AND EQUATIONS

A. Mixture Model

At any time t , we observe, with help of two sensors, two instantaneous mixtures $e_i(t)$ of two unknown zero-mean sources $x_i(t)$, which are assumed statistically independent. It is known that the sources can be separated up to any coefficient; then, diagonal coefficients of the mixture matrix will be supposed equal to one: $m_{ii} = 1$. Denoting M as the unknown mixture matrix with real coefficients, the observations are

$$\begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} = \begin{pmatrix} 1 & m_{12} \\ m_{21} & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}. \quad (1)$$

In the following, we assume the mixture matrix M is regular: $1 - m_{12}m_{21} \neq 0$. We denote the observation (mixture) cross-cumulants:

$$\text{Cum}_{ki}(e_1, e_2) = \text{Cum}(e_1^k(t)e_2^l(t)) = C_{ki} \quad (2)$$

and the unknown moments and cumulants of the sources:

$$p_i = E[x_i^2(t)] \quad (3)$$

$$\gamma_i = E[x_i^4(t)], \quad (4)$$

$$\beta_i = \text{Cum}(x_i^4) = \gamma_i - 3p_i. \quad (5)$$

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