# Performance of power controlled BPSK over a Rayleigh fading channel

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*Indexing terms: Phase shift keying, Fading, Rayleigh channels*

The authors derive an expression for the BER value of the power controlled BPSK over a Rayleigh fading channel, and show that this BER is upper bounded by the BER of the BPSK employing diversity. The derived upper bound is very simple to evaluate. It is shown through computer simulation that this bound is reasonably tight.

*Introduction:* The bit error rate (BER) performance of BPSK over a Rayleigh fading channel depends on the variation of the signal to noise ratio (SNR) at the receiver. Multichannel transmission of the same information-bearing signal over statistically independent fading channels and reception with diversity techniques is one method of reducing the variation [1]. As the number of diversity channels is increased, the BER performance approaches that of an unfaded channel. However, in practice, only a few diversity channels are available because of hardware complexity.

Power control technology was originally introduced to mitigate the near/far problem [2, 3], but it is also effective in reducing the variation caused by the short term Rayleigh fading. The SNR at the receiver of a power controlled system is assumed to be a random variable with lognormal distribution, typically with a variance of 1–2dB [2 – 4].

In this Letter, we derive an expression for the BER value of the power controlled BPSK over a Rayleigh fading channel, and show that this BER is upper bounded by the BER of BPSK employing proper diversity techniques. It is shown through computer simulation that this bound is reasonably tight. This bound, which is shown to be simple to evaluate, provides a useful means of examining the performance of the power controlled BPSK systems and of investigating the relationship between power control and diversity.

*Upper bound of the performance:* Consider a power controlled BPSK signalling over a Rayleigh fading channel. Let $\gamma_b$ be the SNR per bit at the receiver. In deriving the BER of this system, we first evaluate the probability of error conditioned on a fixed $\gamma_b$ and then the conditional probability of error is averaged over the probability density function of $\gamma_b$. When $\gamma_b$ is fixed, the probability of error $P_b$ is given by

$$P_b = \frac{1}{2}\mathrm{erfc}(\sqrt{\gamma_b}) \quad (1)$$

which is equivalent to the BER value of the BPSK over an unfaded channel [1]. If we consider

$$y = 10\log_{10}\gamma_b \quad (2)$$

then eqn. 1 is rewritten as

$$P_b(y) = \frac{1}{2}\mathrm{erfc}(\sqrt{10^{0.1y}}) \quad (3)$$

Since $\gamma_b$ is a random variable having lognormal distribution [3, 4] $y$ is a Gaussian random variable. Then the BER of the power controlled BPSK is given by

$$P_{pc} = \int_{-\infty}^{\infty}\frac{1}{2}\mathrm{erfc}(\sqrt{10^{0.1y}})\hat{p}_{pc}(y)dy \quad (4)$$

where $\hat{p}_{pc}(y)$ is the Gaussian density function of $y$. (The mean and the variance of $y$ will be denoted as $m_{pc}$ and $\sigma_{pc}^2$, respectively.

Now, we consider the performance of BPSK with diversity, which transmits information through $L$ statistically independent fading channels. Each channel is assumed to have Rayleigh-distributed envelope statistics. In such a case, $\gamma_b$ is described as a chi-square random variable whose PDF is [1], p.723

$$p_L(\gamma_b) = \frac{1}{(L-1)!\bar{\gamma}_c^L}\gamma_b^{L-1}e^{-\gamma_b/\bar{\gamma}_c} \quad (5)$$

and $\bar{\gamma}_c(=\gamma_b/L)$ is the average SNR per channel, which is assumed to be identical for all channels. Using $y$ in eqn. 2, PDF in eqn. 5 is rewritten as

$$\hat{p}_L(y) = \frac{C_1}{(L-1)!\bar{\gamma}_c^L}10^{0.1yL}e^{-10^{0.1y}/\bar{\gamma}_c} \quad (6)$$

where $C_1 = \ln 10^{0.1}$. Fig. 1 illustrates $\hat{p}(y)$ for several values of $L$ when $m_{pc}$ = 5dB. Note that the maximum value of $\hat{p}(y)$ is obtained at $y = m_{pc}$; this is easily verified by setting the derivative of $\hat{p}_L(y)$ to zero. The BER of this system with diversity is given by

$$P_L = \int_{-\infty}^{\infty}\frac{1}{2}\mathrm{erfc}(\sqrt{10^{0.1y}})\hat{p}_L(y)dy = \left(\frac{1-\mu}{2}\right)^L\sum_{k=0}^{L-1}\binom{L-1+k}{k}\left(\frac{1+\mu}{2}\right)^k \quad (7)$$

where $\mu = \sqrt{[\bar{\gamma}_c/(1+\bar{\gamma}_c)]}$.

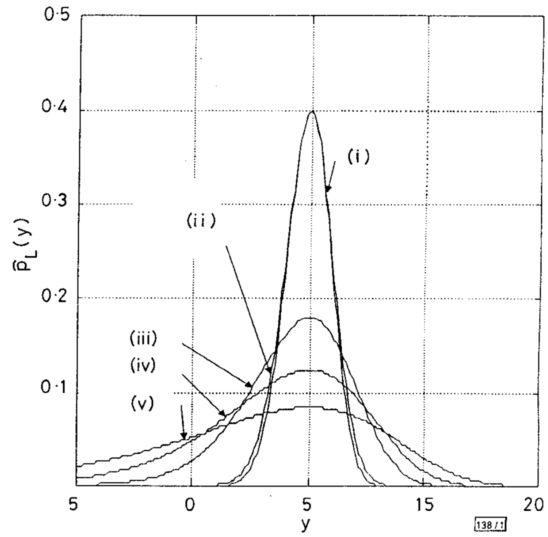


**Fig. 1** *Probability density function* $\hat{p}_L(y)$

(i) $1/\sqrt{[2\pi]}e^{-[(y-5)^2]/2}$
(ii) $L$ = 19
(iii) $L$ = 4
(iv) $L$ = 2
(v) $L$ = 1

The upper bound of $P_{pc}$ in eqn. 4 is obtained by using the following inequality:

$$\int_{-\infty}^{\infty}\mathrm{erfc}(\sqrt{10^{0.1y}})\hat{p}_{pc}(y)dy \le \int_{-\infty}^{\infty}\mathrm{erfc}(\sqrt{10^{0.1y}})\hat{p}_L(y)dy \quad (8)$$

whenever $\hat{p}_{pc}(m_{pc}) = \hat{p}m_{pc})$, that is,

$$\frac{1}{\sigma_{pc}} = \frac{C_2L^L}{(L-1)!}e^{-L} \quad (9)$$

where $C_2 = \sqrt{2\pi}\,C_1$. This inequality in eqn. 8 can be derived by exploiting the fact that $erfc(\sqrt{10^{0.1y}})$ is a monotonically decreasing function of $y$. From eqns. 7 and 8, we can see that the BER $P_{pc}$ in eqn. 4 is bounded by $P_L$, i.e.

$$P_{pc} \le P_L \quad (10)$$

where $L$ is obtained from eqn. 9. This result explicitly shows that the BER of power controlled BPSK is upper bounded by the BER of BPSK with diversity. The procedure for obtaining the upper bound is summarised as follows: first obtain the value of $L$ corresponding to the variance $\sigma_{pc}^2$ of the power controlled BPSK from eqn. 9, then evaluate the bound from eqn. 7.

The relationship between $\sigma_{pc}^2$ and $L$ in eqn. 9 is shown in Fig. 2. For $\sigma_{pc}^2$ = 1, 2 and 3dB, we obtain $L$ = 19, 9 and 7, respectively. The upper bound in eqn. 10 was evaluated for several values of $\sigma_{pc}^2$. The results obtained for $\sigma_{pc}^2$ = 1, 2 and 3dB are illustrated in Fig. 3. To see the tightness of this bound, we obtained the BER values of the power controlled BPSK through computer simulation. The results are also shown in Fig. 3. It is seen that the upper bounds are tight. For example, at $BER = 10^{-2}$, the difference between the bound and the corresponding simulation result is <0.2dB for $\sigma_{pc}^2$ = 2dB.

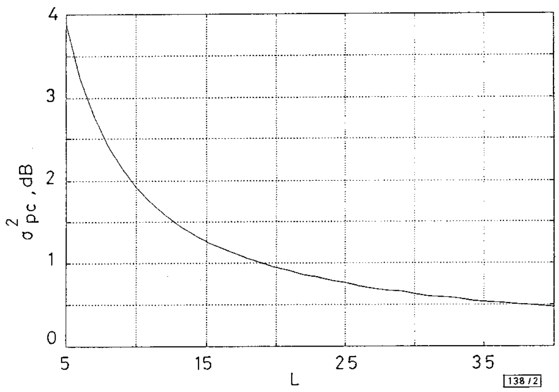


**Fig. 2** *Relation between variance of SNR at receiver and diversity number*

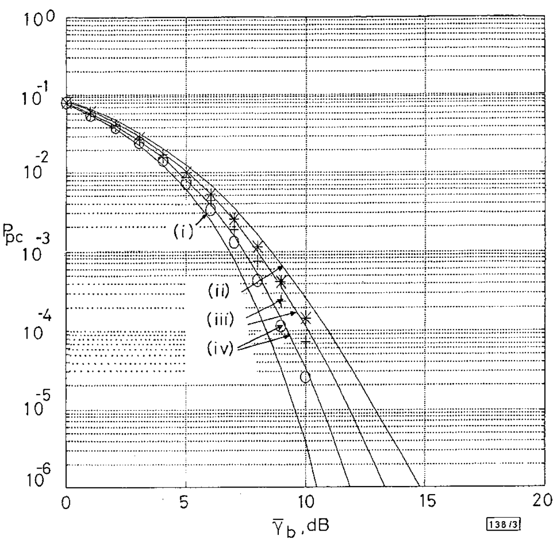


**Fig. 3** *Probability of bit error for power controlled BPSK*

(i) $0.5 erfc\sqrt{\bar{\gamma}_b}$
(ii) $\sigma_{pc}^2 = 3$dB
(iii) $\sigma_{pc}^2 = 2$dB
(iv) $\sigma_{pc}^2 = 1$dB

*Discussion on the relationship between power control and diversity:* Following from the result in eqn. 10, we can explicitly see that the power control techniques, which are mainly used for overcoming the near/far problem, mitigate the effect of shoft term Rayleigh fading. In fact, a power controlled BPSK acts like a BPSK with diversity channels, and a simple power control policy can greatly improve the system performance. For example, a power controlled BPSK with $\sigma_{pc}^2 = 2$dB performs like the BPSK with 9 diversity channels ($L = 9$). In this example, $\sigma_{pc}^2 = 2$dB requires relatively simple power control policy, but the corresponding diversity $L = 9$ is very difficult to implement. Thus, in systems which employ power control methods, the number of diversity channels required to combat fading can be greatly reduced.

*Conclusion:* We have shown that the BER of the power controlled BPSK over a Rayleigh fading channel is upper bounded by the BER of BPSK employing diversity. The derived upper bound is simple to evaluate and reasonably tight. The relationship between power control and diversity was examined based on this bound. It was observed that the number of diversity channels can be greatly reduced by employing power control.



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## Power spectral density of phase noise for a frequency modulated signal plus additive white Gaussian noise after sampling and filtering


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*Indexing terms: Frequency modulation, Demodulation, Statistics for communications*



In an FM(PM)-transmission system the demodulator transforms the complex baseband noise into amplitude and phase noise components. Without modulation, and at moderate to high carrier-to-noise ratio (CNR), the shape of the phase noise power spectral density (PSD) is modelled by the input noise PSD. The authors derive an approximation for modulated signals, which determines the PSD of phase noise near the threshold of FM demodulators.


*Introduction:* Consider a frequency modulated signal with amplitude $a(k)$ and phase $\phi(k)$ disturbed by additive zero mean white Gaussian noise in a complex baseband representation after ideal quadrature sampling and digital lowpass filtering.

$$\underline{z}(k) = a(k)e^{j\phi(k)} + n_c(k) + jn_s(k)$$
$$\phi(k) = 2\pi\Delta F \int_{-\infty}^{kT_s} v(\tau)d\tau \qquad (1)$$

The anti-aliasing filter is assumed to be symmetric around the carrier frequency, so that $n_c(k)$ and $n_s(k)$ are samples of uncorrelated noise components with zero mean and equal variance $\sigma_n^2$ Here $\Delta F$ denotes the frequency deviation and $v(t)$ ($\max|v(t)| = 1$) is the transmitted signal with period $T$, which is for simplification an integer multiple $M$ of the sampling period $T_S$. The information carried by the modulated phase $\phi(k)$ is weakly distorted, if the digital filter is properly designed as a linear phase FIR filter with quadratic amplitude response [1]. Therefore the amplitude $a(k)$ is no longer constant. Rearranging eqn. 1 yields

$$\begin{aligned}\underline{z}(k) &= e^{j\phi(k)}[a(k) + (n_c(k) + jn_s(k))e^{-j\phi(k)}] \\ &= A(k)e^{j\phi(k)}e^{j\arg\{a(k)+N_c(k)+jN_s(k)\}} \\ &= A(k)e^{j(\phi(k)+\phi_n(k))}\end{aligned} \qquad (2)$$

where

$$N_c(k) = n_c(k)\cos\phi(k) + n_s(k)\sin\phi(k)$$
$$N_s(k) = n_s(k)\cos\phi(k) + n_c(k)\sin\phi(k)$$

and

$$A(k) = \sqrt{(a(k) + N_c(k))^2 + N_s(k)^2}$$
$$\phi_n(k) = \arg\{a(k) + N_c(k) + jN_s(k)\} \qquad (3)$$

The amplitude $A(k)$ of the received signal carries no information and is discarded. In the following the PSD of the phase noise process $\phi_n$ is investigated.

*Series expansion of* $\phi_n$: Because taking the argument of the complex baseband signal is a nonlinear operation, there is no simple expression of the PSD available. In such a case, series expansion is a good choice. Assuming that the conditions $a(k) > |N_c(k)|$, $a(k) >$