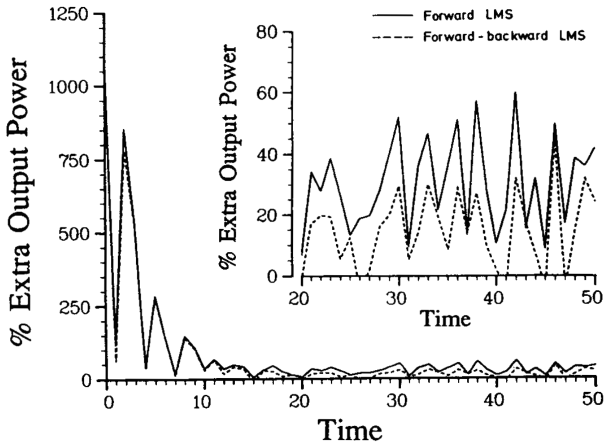

Fig. 3. M versus n plot.

TABLE I
A COMPARISON BETWEEN THE MISADJUSTMENT POWERS OF THE ALGORITHMS

f_0	% Misadjustment	
	forward method	forward-backward method
0.01	31	18
0.1	32	15
0.2	32	17
0.3	33	19
0.4	31	15
0.49	33	19

derive the misadjustment is computed using the exact least squares method using 500 data samples in each run. The misadjustment results listed in Table I are obtained by averaging 100 runs of the excess error power curves after convergence has been achieved.

VII. Conclusions

This paper has presented a new LMS adaptive line enhancer algorithm that makes use of both the forward and backward prediction errors to update the coefficient values. For a given feedback factor, the algorithm is found to converge to the optimal Wiener solution with the same speed and convergence behavior as for the LMS algorithm, but requires about twice the number of multiplications and additions. However, in the situations when the order of the enhancer is at least a few times larger than the number of sinusoids to be enhanced, or when the frequencies of the sinusoids to be enhanced are neither too close to 0 nor 0.5, the misadjustment of the new algorithm is approximately half that of the LMS algorithm. The main characteristics of the algorithm have been verified in numerous simulation studies.

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A Study of Convex/Concave Edges and Edge-Enhancing Operators Based on the Laplacian

YONG HOON LEE AND SOON YOUNG PARK

***Abstract* —We consider edge-enhancing operators employing a modification of the Laplacian called the order statistic (OS) Laplacian. The edge-enhancing operators are evaluated for their performance on the convex/concave (C/C) edge, which is a useful blurred edge model, and on white Gaussian noise input signals. It is shown that the operators employing the OS Laplacian are much less sensitive to noise than the edge-enhancing operator employing the Laplacian, while the edge-enhancing characteristics of the former are comparable to those of the latter. One set of images processed by these operators is presented to illustrate the performance characteristics of these operators.**

I. Introduction

In many cases, the restoration of blurred images has been carried out through linear deconvolution. While linear deconvolution procedures are very effective in deblurring, they are rather hard to implement because sufficient knowledge of a model representing the blur is required. Image sharpening or edge-enhancing techniques are simple image enhancement tools counteracting blur without knowledge about the blur. One of the edge-enhancing techniques, and probably the most widely used one, is $F-\nabla^2F$, where F is an image and ∇^2F is the

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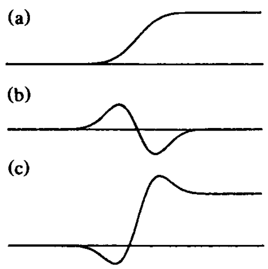

Fig. 1. (a) A C/C edge, $F(x)$. (b) d^2F/dx^2. (c) $F - d^2F/dx^2$.

Laplacian operator [1].[1] Because the edge-enhancing techniques that we shall consider are based on $\nabla^2 F$, the reason why $F - \nabla^2 F$ has edge-enhancing effect is worth describing here.

Consider a typical one-dimensional (1-D) blurred edge shown in Fig. 1(a). The increasing edge curve is bending upward at the lower side of the edge, and bending downward at the higher side of the edge. It is important to note that, in general, the second derivative of a curve bending upward is positive, while that of a curve bending downward is negative. Thus $F - \nabla^2 F$ is generally less than F at the lower side of the edge, and larger than F at the higher side. This results in edge enhancement. The second derivative of the edge in Fig. 1(a) and the enhanced edge are shown in Fig. 1(b) and (c), respectively. In general, a curve bending upward is convex, and a curve bending downward is concave. In many practical situations, an increasing edge is composed of increasing convex part followed by a concave one, while a decreasing edge is composed of decreasing concave part followed by a convex one. Such edges are called the convex/concave (C/C) edges [2]. In essence, $F - \nabla^2 F$ implicitly assumes that the edge to be enhanced is C/C.

The major difficulty in applying the Laplacian to images is its sensitivity to noise. To reduce the noise sensitivity, the Laplacian is often used after some pre-filtering. For example, edge-preserving filters such as the median filter [1], [3] may be used before applying the Laplacian. Alternatively, some modifications of the Laplacian may be used [1]. One such operator, which we call the order statistic (OS) Laplacian, is an operator proportional to the difference between the average and the median, computed over the same neighborhood of the given point. The modified Laplacian operators are less sensitive to noise than the Laplacian.

Recently, Lee and Fam [2] introduced a 1-D discrete C/C edge model, and an edge-enhancing operator called the comparison and selection (CS) filter was derived based on this edge model. It has been observed that CS filters can enhance edges while suppressing noise. We can see that the CS filter is, in fact, an operator employing the OS Laplacian. This is observed from the following definition of the CS filter. If we let $\{F(.)\}$ and $\{G(.)\}$ be the input and output, respectively, of the CS filter, then the output at (i,j) is given by

$$G(i,j) = \begin{cases} F(N+1-J;i,j), & \text{if } A(i,j) \geq M(i,j) \\ F(N+1+J;i,j), & \text{otherwise} \end{cases} \quad (1)$$

[1]The image F is either continuous or discrete. When it is necessary to distinguish a continuous image from a discrete one, continuous and discrete images are denoted by F_a and F, respectively. When F is a two-dimensional (2-D) continuous image having second derivatives, $\nabla^2 F = \partial^2 F/\partial x^2 + \partial^2 F/\partial y^2$. When F is discrete, $\nabla^2 F$ is a discrete analog of the continuous Laplacian. In addition, if F is 1-D, $\nabla^2 F = d^2F/dx^2$.

where $F(k;i,j)$ is the kth smallest sample among the values within a window centered at (i,j), $2N+1$ is the size of the window, J is an integer between one and N, and $A(i,j)$ and $M(i,j)$, respectively, are the sample average and sample median of the data inside the window. The CS filter selects one of the sample values inside the window depending on the sign of the OS Laplacian.

In this paper, the characteristics of the edge-enhancing operators, which are based on the Laplacian and OS Laplacian operators, are analyzed. Their edge-enhancing characteristics are examined through a deterministic analysis that shows the effects of the edge-enhancing operators on C/C edges. The noise sensitivities of the operators are analyzed statistically. These analyses provide further understanding of these operators and also show that the OS Laplacian has some desirable properties for image enhancement.

The organization of this paper is as follows. In Section II, the OS Laplacian is defined. Sections III and IV give the deterministic and statistical analysis, respectively. Finally, in Section V, some experimental results are presented.

II. The OS Laplacian

The digital Laplacian of an image at point (i,j) is commonly implemented by [1]

$$\nabla^2 F(i,j) = L[A(i,j) - F(i,j)] \quad (2)$$

where L is the number of samples inside a window centered at (i,j) and $A(i,j)$ is the average of the samples inside the window. The OS Laplacian, denoted by $\text{OS}\nabla^2 F$, is defined by

$$\text{OS}\nabla^2 F(i,j) = L[A(i,j) - M(i,j)] \quad (3)$$

where $M(i,j)$ is the sample median of the values inside the window. Clearly, the OS Laplacian is equal to the digital Laplacian whenever the center value of the window, $F(i,j)$, equals the median, $M(i,j)$. We can see that the OS Laplacian is a linear combination of the ordered values inside the window. So the OS Laplacian is a special case of the OS filter introduced by Bovik *et al.* [4].

The window centered at the point (i,j), say W_{ij}, is defined in terms of neighborhood pixel locations. For example, the $(2N+1)\times(2N+1)$ square-shaped window centered at (i,j) is given by $W_{ij} = \{(m,n) | i-N \leq m \leq i+N, j-N \leq n \leq j+N\}$. The size of a window is the total number of pixel locations inside it. We always assume that the window W_{ij} includes the center point (i,j) and is symmetric with respect to the center, that is, $(m,n) \in W_{ij}$ implies $(2i-m,n) \in W_{ij}$, $(m,2j-n) \in W_{ij}$, and $(2i-m,2j-n) \in W_{ij}$. For such a window, an odd window size is guaranteed, and thus the median value can always be selected.

The edge-enhancing operators that we shall consider include $F - \nabla^2 F$, $M - \nabla^2 M$, which is median filtering, followed by $F - \nabla^2 F$, $F - \text{OS}\nabla^2 F$, and CS filters. It will be shown that the operators employing the OS Laplacian are much less sensitive to noise than $F - \nabla^2 F$, while the edge-enhancing characteristics of the former are comparable to that of the latter.

III. Enhancement of C/C Edges

A natural first step in analyzing the edge-enhancing characteristics of the operators defined in the previous section is to examine the effect of these operators on blurred edges. In this section, we apply the operators to 2-D C/C edges, which is a useful blurred edge model. Before describing the edge-enhancing properties, 2-D C/C edges are defined and their characteristics are studied briefly.

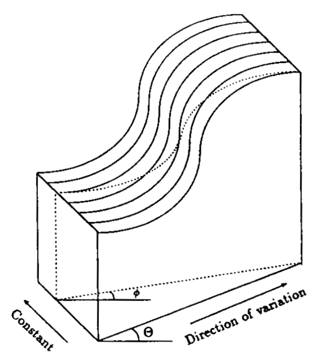

Fig. 2. A 2-D C/C edge with orientation Θ. The dotted line shows the profile taken in orientation ϕ.

A. 2-D C/C Edges

The 2-D edge is defined as a surface that is constant in one direction; its profile taken in the edge orientation, which is the direction perpendicular to the direction of invariance, is a 1-D edge, as shown in Fig. 2 (see Appendix for the definition of a 1-D edge). The edge orientation, which is the angle measured from the positive half of the x-axis, is represented by Θ. For simplicity, our results are presented for $0 \leq \Theta \leq \pi/2$, but they can be extended to Θ outside the range in a straightforward manner. Next we present formal definitions of 2-D continuous and discrete C/C edges.

Let $F_\Theta(x,y)$, $-\infty \leq x, y \leq \infty$, be a 2-D continuous edge with orientation Θ. We define a shifted and rotated coordinated system, the $(x_{\phi bc}, y_{\phi bc})$ system, as $x_{\phi bc} = (x-b)\cos\phi + (y-c)\sin\phi$, and $y_{\phi bc} = -(x-b)\sin\phi + (y-c)\cos\phi$, where b and c represent the amount of shifting in x- and y-directions, respectively, and ϕ is the angle of rotation, $0 \leq \phi \leq 2\pi$. It is seen that the orientation of the edge $F_\Theta(x,y)$ is parallel to the $x_{\Theta bc}$-axis and normal to $y_{\Theta bc}$-axis. The profile of $F_\Theta(x,y)$ taken in the orientation ϕ at the point (b,c) can be expressed as $P_F(x_\phi) = F_\Theta(x_\phi \cos\phi + b, x_\phi \sin\hat{\phi} + c)$ where we write x_ϕ instead of $x_{\phi bc}$ to simplify the notation. Now the profile taken in the edge orientation is written as $P_F(x_\Theta)$, and the profile taken in the direction of invariance is written as $P_F(x_\phi)$ with $\phi = \Theta + \pi/2$. The 2-D edge with orientation Θ is defined formally as follows:

Definition 1: A real-valued function $F_\Theta(x,y)$, $-\infty < x, y < \infty$, is a 2-D edge with orientation Θ if the profile $P_F(x_\phi)$ with $\phi = \Theta + \pi/2$ is constant, and the profile $P_F(x_\Theta)$ is a 1-D edge for any constants b and c. In addition, if $P_F(x_\Theta)$ is a 1-D C/C edge, then $F_\Theta(x,y)$ is a 2-D C/C edge with orientation Θ.

We now observe that a profile of a 2-D C/C edge taken in an arbitrary direction is a 1-D C/C edge if it is not constant (see, again, Fig. 2).

Property 1: A 2-D edge, $F_\Theta(x,y)$, with orientation Θ, is C/C if and only if the profile $P_F(x_\phi)$ is 1-D C/C for any ϕ, $0 \leq \phi \leq 2\pi$, and $\phi \neq \Theta \pm \pi/2$.

Proof: Suppose that $F_\Theta(x,y)$ is C/C. Then $P_F(x_\Theta)$ is a 1-D C/C edge. Consider $P_F(x_\phi)$ with a ϕ, $0 \leq \phi \leq 2\pi$. Assume that $P_F(x_\Theta)$ and $P_F(x_\phi)$ are taken at the origin ($b = c = 0$). Since $F(x,y)$ is constant in the direction $\Theta + \pi/2$, we get $P_F(x_\phi) = P_F(x_\Theta)$ when $x_\Theta = tx_\phi$ with $t = \sin\Theta\sin\phi + \cos\Theta\cos\phi$. Hence $P_F(x_\phi)$ is convex if $x_\phi \leq x_{\Theta 0}/t$, and concave otherwise, where $x_{\Theta 0}$ is the inflection point of the 1-D C/C edge $P_F(x_\Theta)$. Similarly, we can show the case where the profiles are taken at an arbitrary point. The proof of the converse is trivial.

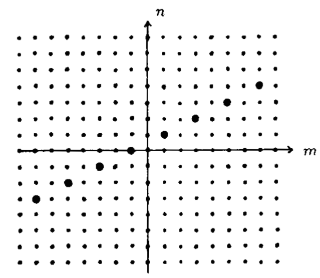

Fig. 3. Integer pairs $(m_\phi h_\phi + i, m_\phi v_\phi + j)$ with $v_\phi = 1$, $h_\phi = 2$, and $i = j = 1$.

In Section III-B we shall see that Property 1 and its discrete version (Property 2) are useful in examining the characteristics of edge-enhancing operators.

Unlike the continuous case, the 2-D discrete edge cannot be defined for all orientations. This is because the orientation in discrete domain is given by the angle

$$\Theta = \tan^{-1}(v_\Theta / h_\Theta) \tag{4}$$

where v_Θ and h_Θ must be integers. We always assume that the orientation of the 2-D discrete edge is given by (4). Let the 2-D discrete edge with orientation Θ be described by the array $F_\Theta(m,n)$, $-\infty < m, n < \infty$. A shifted and rotated discrete coordinated system, the (m_ϕ, n_ϕ) system, is defined as $m_{\phi ij} = (m-i)v_\phi + (n-j)h_\phi$ and $n_{\phi ij} = -(m-i)h_\phi + (n-j)v_\phi$, where v_ϕ and h_ϕ are the smallest positive integers satisfying $\tan\phi = v_\phi / h_\phi$ (e.g., if $\phi = 45$, then $v_\phi = h_\phi = 1$), and i, j are integers representing the amount of shifting. The profile of $F_\Theta(m,n)$, taken in the orientation ϕ at the point (i,j), can be expressed as $P_F(m_\phi) = F_\Theta(m_\phi h_\phi + i, m_\phi v_\phi + j)$, where ij is dropped from $m_{\phi ij}$ to simplify the notation. Fig. 3 illustrates the integer pairs $(m_\phi h_\phi + i, m_\phi v_\phi + j)$ for the special case $v_\phi = 1$, $h_\phi = 2$, and $i = j = 1$.

The 2-D discrete edge is derived from the 2-D continuous edge by periodic sampling.

Definition 2: If $F_{a\Theta}(x,y)$ represents a continuous edge with orientation Θ where Θ is given by (4), then the discrete edge $F_\Theta(m,n)$ with orientation Θ is given by $F_\Theta(m,n) = F_{a\Theta}(mT, nT)$ where m and n are integers, $-\infty < m, n < \infty$. The discrete edge is C/C whenever the continuous edge is C/C.

The following property is a discrete version of Property 1.

Property 2: Let $F_\Theta(m,n)$ be a 2-D discrete C/C edge with orientation Θ. Then the profile $P_F(m_\phi)$ taken in any orientation ϕ, $0 \leq \phi \leq 2\pi$, $\phi \neq \Theta \pm \pi/2$, is a 1-D discrete C/C edge.

Since this is a direct consequence of Property 1, the proof is omitted.

B. C/C Edge Enhancement

For a 1-D C/C edge, due to its monotonicity, it is obvious that the Laplacian and the OS Laplacian of the edge are the same. This observation can be extended to 2-D C/C edges as follows: Let $F(m,n)$ be a 2-D C/C edge and $L_{ij}(\phi)$ be the line passing through the points (i,j) and $(i + h_\phi, j + v_\phi)$, where $\tan\phi = v_\phi / h_\phi$. Since the profile taken in the orientation ϕ is either 1-D C/C or constant from Property 2, we get Median $\{F(m,n) | (m,n) \in L_{ij}(\phi) \cap W_{ij}\} = F(i,j)$ for each ϕ. Thus the median $M(i,j)$ selected from W_{ij} is equal to $F(i,j)$, which is the center value of the window [5]. This indicates that the 2-D C/C edge is invariant to 2-D median filtering with a symmetric window W_{ij}, and thus $\nabla^2 F = \mathrm{OS}\nabla^2 F$ and $F - \nabla^2 F = M - \nabla^2 M = F - \mathrm{OS}\nabla^2 F$. Therefore, in fact, the analysis presented below, which shows that $F - \mathrm{OS}\nabla^2 F$ enhances C/C edges, gives another rationale to the well-known operator $F - \nabla^2 F$.

The edge-enhancing operators, such as $F - \mathrm{OS}\nabla^2 F$ and CS filters, enhance C/C edges if the OS Laplacian is positive in convex regions and is negative in concave regions. In the rest of this section, we will study the 2-D C/C edge-enhancing properties of these operators by examining the sign of the OS Laplacian.

Property 3: Suppose that a symmetric window W_{ij} moves over a 2-D C/C edge. Then $A(i,j) \geq M(i,j)$ if all samples inside W_{ij} are from the convex region, and $A(i,j) \leq M(i,j)$ if all of the samples are from the concave region.

Proof: Since $M(i,j) = M_\phi(i,j) = F(i,j)$, we get $A(i,j) - M(i,j) = \sum_\phi n_\phi [A_\phi(i,j) - M_\phi(i,j)]/L$, where $M_\phi(i,j) = \text{Median}\{F(m,n) | (m,n) \in L_{ij}(\phi) \cap W_{ij}\}$, $A_\phi(i,j) = \text{Average}\{F(m,n) | (m,n) \in L_{ij}(\phi) \cap W_{ij}\}$, n_ϕ is the number of samples in $L_{ij}(\phi) \cap W_{ij}$, and ϕ ranges from 0 to π. Due to Property 2 in Section III-A and property 2 in [2], $A_\phi(i,j) \geq M_\phi(i,j)$ in the convex region, and the inequality is reversed in the concave region for each ϕ. This completes the proof.

This property implies that the OS Laplacian of a 2-D C/C edge is nonnegative if all the samples within the window are from the convex region, and is nonpositive if all the samples are from the concave region. Therefore, we can generally say that the 2-D edge-enhancing operators employing the OS Laplacian can enhance 2-D C/C edges. The following property considers the cases where the window is located in the transition region, in which some samples in the window are from the convex region and the rest are from the concave region.

Property 4: Let $F_\Theta(m,n)$ be a 2-D discrete C/C edge with orientation Θ and a symmetric window W_{ij} scans $F_\Theta(m,n)$ in the horizontal direction. If either W_{ij} is square-shaped and $\Theta = 0$, or W_{ij} is cross-shaped and $\Theta = 0$ or $\pi/4$, then there exists an integer k_j for which $A(i,j) \geq M(i,j)$ if $i \leq k_j$, and $A(i,j) \leq M(i,j)$ if $i > k_j$ for each j.

Proof: If the $(2N+1)\times(2N+1)$ square-shaped window is employed and $\Theta = 0$, then $A(i,j) - M(i,j) = A_0(i,j) - M_0(i,j)$ where $A_0(i,j)$ and $M_0(i,j)$ are given with $\phi = 0$. If the $(2N+1)\times(2N+1)$ cross-shaped window is employed, then $A(i,j) - M(i,j) = C[A_0(i,j) - M_0(i,j)]$, with $C = (2N+1)/(4N+1)$ if $\Theta = 0$, and $C = 2(2N+1)/(4N+1)$ if $\Theta = \pi/4$. Hence for each case the 2-D problem is reduced to a 1-D problem and the property holds by Property 2 in Section III-A and lemma 1 in [2].

It should be noted that the point k_j at which the sign of the OS Laplacian is changed is unique for each j. This prevents the edge-enhancing operators from causing some oscillations around the transition region.

TABLE I
OUTPUT VARIANCES OF THE OPERATORS WHEN THE WINDOW SIZE IS 3×3 AND THE INPUT VARIANCE IS ONE

Type of operator		Output variance: square window	Output variance: cross window
$F-\nabla^2 F$		89.0	29.0
$M-\nabla^2 M$		6.648	5.570
$F-OS\nabla^2 F$		5.456	3.170
CS	J = 1	0.281	0.548
	J = 2	0.505	1.414
	J = 3	0.955	
	J = 4	2.135	

A property similar to Property 4 holds when the window scans the edge in the vertical direction.

IV. Noise Sensitivity

In this section, we study the noise sensitivity properties of $F - \nabla^2 F$, $M - \nabla^2 M$, $F - \mathrm{OS}\nabla^2 F$, and CS filters by comparing their output variances when the input signal $F(i,j)$ is independently and identically distributed with the Gaussian density function. The procedures for obtaining the output variances of $M - \nabla^2 M$ and $F - \mathrm{OS}\nabla^2 F$ follow. (It is trivial to obtain the output variance of $F - \nabla^2 F$, and that of the CS filter has been obtained in [6].) If we denote the results of $M - \nabla^2 M$ and $F - \mathrm{OS}\nabla^2 F$ by $G_M(i,j)$ and $G_O(i,j)$, respectively, then they are given by

$$G_M(i,j) = M(i,j) + LM(i,j) - \sum_{(i-k,j-m)\in W_{ij}} M(i-k,j-m) \tag{5}$$

$$G_O(i,j) = F(i,j) + LM(i,j) - \sum_{(i-k,j-m)\in W_{ij}} F(i-k,j-m). \tag{6}$$

It is not difficult to see that $M - \nabla^2 M$ and $F - \mathrm{OS}\nabla^2 F$ are unbiased estimators of the mean if the input density is symmetric with respect to the origin. (The unbiasedness of the CS filter is shown in [6].) Therefore, when the input has zero mean, the variances of $M - \nabla^2 M$ and $F - \mathrm{OS}\nabla^2 F$ denoted by σ_M^2 and σ_O^2, respectively, are given by

$$\begin{aligned}\sigma_M^2 &= E\left[G_M^2(i,j)\right] \\ &= (L+1)^2 E\left[M^2(i,j)\right] - 2(L+1) \sum_{(i-k,j-m)\in W_{ij}} E[M(i,j)M(i-k,j-m)] \\ &\quad + \sum_{(i-k,j-m)\in W_{ij}} \sum_{(i-s,j-t)\in W_{ij}} E[M(i-k,j-m)M(i-s,j-t)]\end{aligned} \tag{7}$$

and

$$\begin{aligned}\sigma_O^2 &= E\left[G_O^2(i,j)\right] \\ &= (L-1)\sigma^2 + L^2 E\left[M^2(i,j)\right] \\ &\quad - 2L(L-1)E[M(i,j)F(i,j)]\end{aligned} \tag{8}$$

where σ^2 is the input variance.

The evaluation of σ_M^2 and σ_O^2 requires the joint probability density function of median values that can be obtained by using the formulas in [7]. Using (7) and (8), we numerically evaluate σ_M^2 and σ_O^2 for the Gaussian input with zero mean and variance one. The results associated with the 3×3 square- and cross-shaped windows are tabulated in Table I. For comparison, the

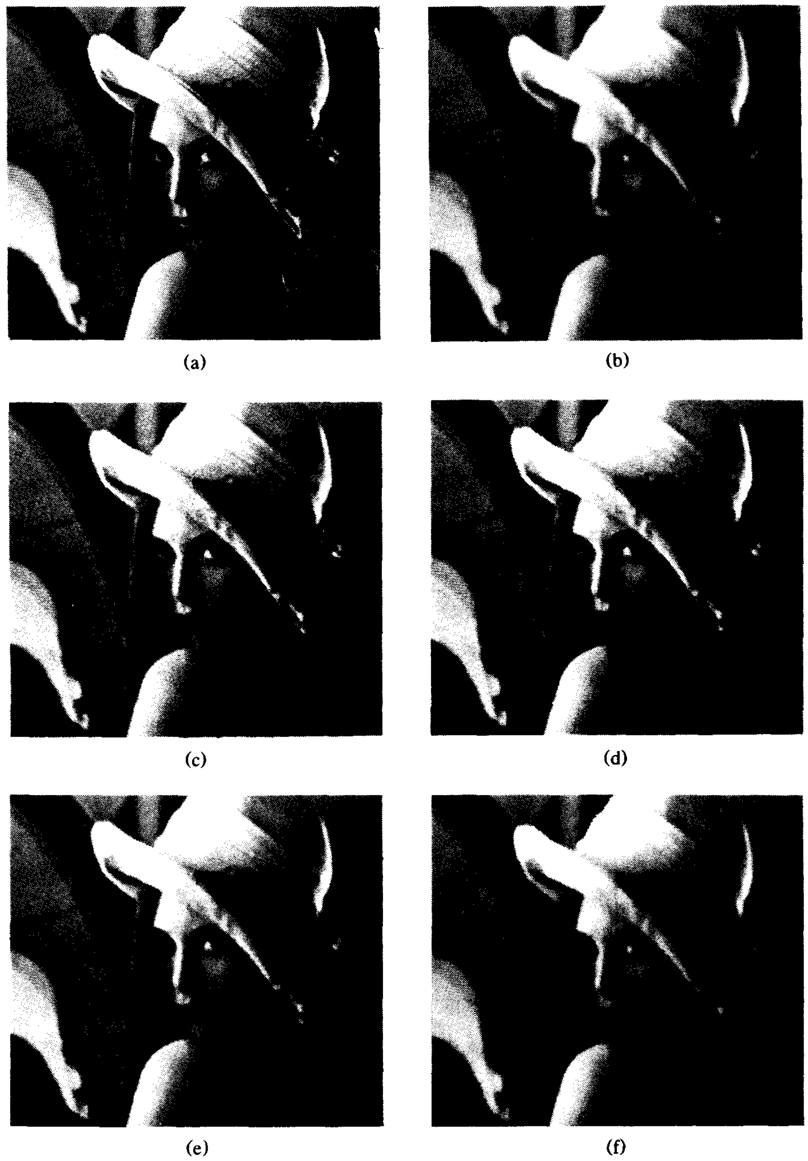

Fig. 4. (a) The original image. (b) The image degraded by blurring and additive Gaussian noise with variance 5. (c) The noisy blurred image enhanced by $F-\nabla^2 F$. (d) The noisy blurred image enhanced by $M-\nabla^2 M$. (e) The noisy blurred image enhanced by $F-\mathrm{OS}\nabla^2 F$. (f) The noisy blurred image enhanced by CS, $J=1$.

output variance of $F-\nabla^2 F$ and CS filters are also shown. The results show that only the CS filter can reduce noise components, while the others amplify them. As expected, $F-\nabla^2 F$ severely amplifies noise, while $M-\nabla^2 M$ and $F-\mathrm{OS}\nabla^2 F$ are much less sensitive to noise than $F-\nabla^2 F$. Between $M-\nabla^2 M$ and $F-\mathrm{OS}\nabla^2 F$, the latter is slightly less sensitive to noise than the former.

Although $M-\nabla^2 M$ and $F-\mathrm{OS}\nabla^2 F$ amplify noise, sometimes they can produce more subjectively pleasing images than the CS filter. This is because $M-\nabla^2 M$ and $F-\mathrm{OS}\nabla^2 F$ accentuate edges, as will be seen in the next section.

V. Experimental Results

In this section, edge-enhancing operators are applied to a noisy blurred image to assess their performance. The images under consideration consist of 256×256 pixels with eight bits of resolution. Throughout this section, we consider edge-enhancing operators with 3×3 cross-shaped windows.

Fig. 4(a) shows the original image. The image was blurred by passing it twice through a 3×3 mean filter, which replaces the center value of each 3×3 square-shaped window with the average of values in the window. Zero-mean white Gaussian noise with variance 5 was added to the blurred image, and the result is shown in Fig. 4(b). Fig. 4(c)–(f), respectively, exhibit the results of $F-\nabla^2 F$, $M-\nabla^2 M$, $F-\mathrm{OS}\nabla^2 F$, and CS filtering with $J=1$ of the noisy blurred image. As expected, $F-\nabla^2 F$ amplified the noise. It appears that the edge-enhancing characteristics of $M-\nabla^2 M$ and $F-\mathrm{OS}\nabla^2 F$ are comparable to those of $F-\nabla^2 F$, while they amplified noise less than $F-\nabla^2 F$. Therefore, $M-\nabla^2 M$ and $F-\mathrm{OS}\nabla^2 F$ are useful alternatives to $F-\nabla^2 F$ in

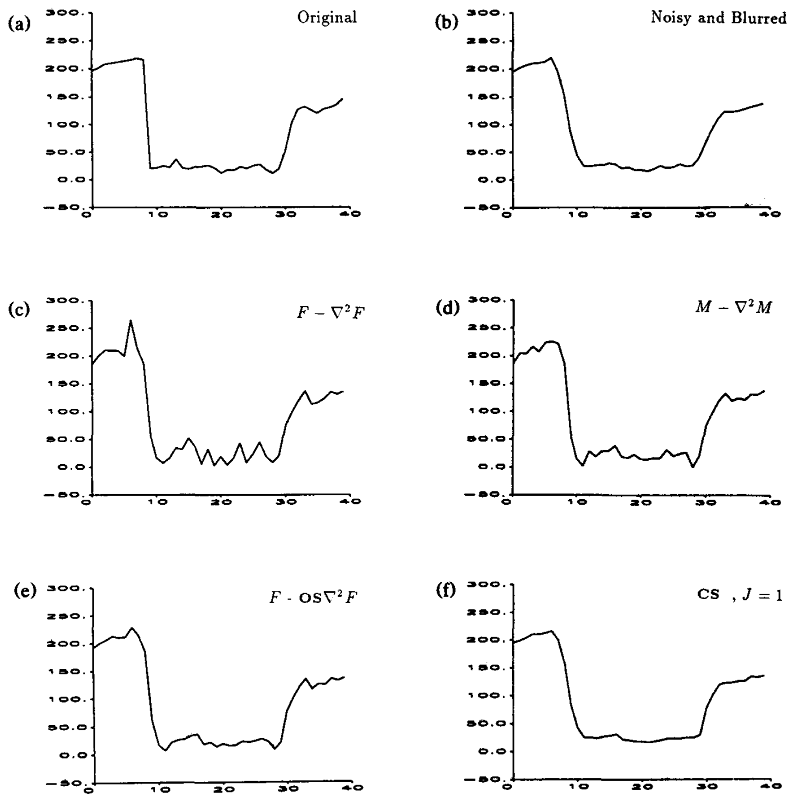

Fig. 5(a)–(f). Profiles taken from Fig. 4(a)–(f), respectively.

image enhancement. Visually, the results of $M-\nabla^2 M$ and $F-\mathrm{OS}\nabla^2 F$ are preferred to those of CS filtering. The CS filter that enhanced edges while suppressing noise may be preferred to the others as a prefilter before edge detection [2], [6].

In order to show better the edge-enhancing properties and noise sensitivity of the operators, Fig. 5(a)–(f) compare a profile of the original image with its enhanced versions. $F-\nabla^2 F$, $M-\nabla^2 M$, and $F-\mathrm{OS}\nabla^2 F$ accentuated edges by introducing undershooting and overshooting, respectively, at the lower sides and at the higher sides of the edges. On the other hand, the CS filter tended to produce piecewise constant type edges. As expected, the CS filter did not amplify the noise, while it was amplified severely by $F-\nabla^2 F$. Finally, we can see that $M-\nabla^2 M$ and $F-\mathrm{OS}\nabla^2 F$ slightly amplified the noise.

VI. Conclusion

The C/C edges have been defined and characterized. Edge-enhancing operators employing the OS Laplacian were analyzed. It was shown that these operators enhance 2-D C/C edges. The output variances of the edge-enhancing operators were evaluated numerically when the input was white Gaussian. Some experimental results were presented to illustrate the performance characteristics of the edge-enhancing operators.

In this paper, C/C edges were used in characterizing the performance of edge-enhancing operators. The C/C edge, which is a practical edge model, should be useful for other image processing applications such as edge detection. Examining edge detectors by using C/C edges should give more insight into their performance than by using step or ramp edge models.

Appendix
One-Dimensional C/C Edges

First, nondecreasing edges are defined formally.

Definition A1: A nonnegative real-valued function $F(x)$ defined in $(-\infty,\infty)$ is a nondecreasing edge if (1) $F(x)$ is a nondecreasing function with $\lim_{x\to-\infty} F(x)=0$ and $\lim_{x\to\infty} F(x)=c$ for some positive constant c; (2) $F(x)$ is either continuous or piecewise continuous; and (3) $F(x)$ has derivative $F'(x)$ except at some isolated points, i.e., $F(x)$ is piecewise differentiable.

The C/C edge is defined as follows.

Definition A2: A continuous nondecreasing edge $F(x)$ is C/C if $F(x)$ is convex if $x \le x_0$, and concave otherwise, for some x_0.

The characteristics of 1-D continuous and discrete C/C edges may be found in [2] and [8].

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A Minimax Approach to the Nonlinear Equalizability Problem

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Abstract —**Based on a minimax criterion, we define the concept of equalizability for a nonlinear, discrete-time communication channel. Sufficient conditions for a channel to be equalizable, via a finite memory equalizer, are also derived.**

I. Introduction

The goal of digital communications is to produce a system with the best performance features, such as low power consumption and high information density, while minimizing the probability of error. In seeking more desirable channel features, system degradations due to noise and distortion must be lowered to maintain a constant probability of error. It is for this reason that the equalization of nonlinear communication channels has emerged as an important problem in the field of digital communications [1]. Nonlinear equalization refers to the elimination of the intersymbol interference due to distortion in communication channels via the use of filters, called "equalizers," which follow the channel. Although in practice the proper equalizer must be selected from some chosen model set, equalizability addresses only the question of whether a channel can be equalized within a particular model set. In this paper, we present a study of equalizability with respect to the class of finite memory equalizers. The concept of equalizability is defined using a minimax type criterion.

II. Notation and Definitions

Digital communication channels are viewed as deterministic, discrete-time operators. The following conventions will be adhered to, concerning sequences and operators on these sequences.

a) A bisequence is a map from $\mathbf{Z}$ into $\mathbb{R}$ (here $\mathbf{Z}$ denotes the integers).
b) $(l_\infty, \|\cdot\|_\infty)$ denotes the normed space of all bounded bisequences.
c) $l_\infty(M)$ denotes the space of all bisequences $\boldsymbol{u}$ such that $\|\boldsymbol{u}\|_\infty \leqslant M$, $M \in \mathbb{R}$.
d) If $G: l_\infty \to l_\infty$, and $\boldsymbol{u} \in l_\infty$, then $G(\boldsymbol{u})(n)$ denotes the value of the "output" bisequence at time n due to the "input" bisequence $\boldsymbol{u}$, while $G(\cdot)(n): l_\infty \to \mathbb{R}$ denotes the corresponding map.

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We introduce the following operators (see [4]): For $N \in \mathbf{Z}$, the *truncation* $T_N: l_\infty \to l_\infty$ is defined by

$$T_N(\boldsymbol{w})(l) = \begin{cases} w(l), & \text{if } l \leqslant N \\ 0, & \text{otherwise} \end{cases}$$

while the *shift* operator $\Delta_N: l_\infty \to l_\infty$ is defined by

$$\Delta_N(\boldsymbol{w})(l) = w(1-N), \qquad \boldsymbol{w} \in l_\infty. \tag{2.1}$$

Finally, an operator $G: l_\infty \to l_\infty$ is called a *finite memory* (FM) operator if there exist $L, L' \in \mathbf{Z}$ such that

$$G(\boldsymbol{v})(n) = G(T_{n+L'}(\boldsymbol{v}) - T_{n-L}(\boldsymbol{v}))(n), \quad \text{for all } \boldsymbol{v} \in l_\infty(P),\ n \in \mathbf{Z}. \tag{2.2}$$

We now define the concepts of a communication channel and an equalizer.

Definition 2.1: A (discrete-time) *communication channel* consists of an operator $C: l_\infty(M) \to l_\infty(P)$, for some M and $P \in \mathbb{R}$, such that:

i) $T_N \circ C \circ T_N = T_N \circ C$, for all $N \in \mathbf{Z}$ (causality)
ii) $\Delta_N \circ C = C \circ \Delta_N$, for all $N \in \mathbf{Z}$ (time-invariance).

Definition 2.2: An *equalizer E* associated with a communication channel C is a map from $l_\infty(P)$ into $l_\infty(Q)$, which satisfies i) and ii) of Definition 2.1.

For clarity, throughout this paper, the channel input bisequence is denoted by $\boldsymbol{u} \in l_\infty(M)$ and the channel output bisequence (and hence the equalizer input bisequence) is denoted by $\boldsymbol{x} \in l_\infty(P)$. Due to time-invariance, a channel is uniquely defined by merely specifying $C(\cdot)(n)$, for some $n \in \mathbf{Z}$.

A metric on the space of all causal, time-invariant operators mapping $l_\infty(M)$ into $l_\infty(Q)$ that induces a notion of distance applicable to the study of equalizability of digital communication channels is defined by

$$d(G_1, G_2) = \sup_{\boldsymbol{v} \in l_\infty(M)} \|G_1(\boldsymbol{v}) - G_2(\boldsymbol{v})\|_\infty. \tag{2.3}$$

Observe that as a result of time-invariance, and under the hypothesis that the input space is closed under the shift operator, for every $n_0 \in \mathbf{Z}$, we have

$$d(G_1, G_2) = \sup_{\boldsymbol{v} \in l_\infty(M)} |G_1(\boldsymbol{v})(n_0) - G_2(\boldsymbol{v})(n_0)|.$$

Definition 2.3: A channel C is said to be *equalizable* if, for every $\epsilon > 0$, there exists an equalizer E and an $N \in \mathbf{Z}$ such that $d(E \circ C, \Delta_N) < \epsilon$. Also, if Θ is a given subset of the set of all equalizers, a channel C is said to be *Θ-equalizable* if the above condition is true for some $E \in \Theta$.

III. Main Results

Let Θ_{FM} denote the space of finite memory equalizers. We are going to restrict our attention to the study of Θ_{FM}-equalizability. This is analogous to the restriction of the linear equalizability problem to finite impulse response (FIR) filters. The closure of Θ_{FM}, relative to the metric d defined in (2.3), is a fairly large class of nonlinear operators. In particular, it contains those Hammerstein channels [2] whose memoryless nonlinearity is a bounded map. Our main results are summarized in the Theorems 3.1 and 3.2. In the linear case, these theorems reduce to the adaptive equalization results of Lucky [3]. In this section, $\mathscr{B}^n(0,r) \subset \mathbb{R}^n$ denotes the closed ball of radius r, centered at the origin, relative to the l_∞ norm.

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