Spectral Performance Characterizations of some Generalized Median Filters

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ABSTRACT: *Output correlations and power spectral densities are obtained for two types of generalized median filters: the L- and M-filters, for a sequence of independent, identically distributed inputs and also for a simple first-order Markov input sequence. The results show the ability of generalized median filters to flexibly achieve performance compromises between edge preservation and impulsive noise rejection, and Gaussian noise suppression.*

I. Introduction

Recently, Generalized Median Filter (GMF) techniques have been developed for applications where linear filtering is inadequate (**1**–**3**). If a desired discrete-time signal with sharp edges is corrupted by noise, then linear filters designed to reduce the noise will also smooth out signal edges. Additionally, impulsive noise components are not well suppressed by linear filtering. In such cases, some form of nonlinear or adaptive filter such as a GMF is preferable.

The nonlinear nature of GMFs makes filter design and performance characterization difficult. Simple median filtering has been partially characterized deterministically and statistically by several researchers (**4**–**11**). One useful characterization has been the output spectral performance for white noise inputs. While a GMF's nonlinearity precludes direct generalization of the white noise results to other input cases, as is possible with linear filters, such results nevertheless provide useful comparisons against linear filter performances. With careful interpretation, the results can also suggest performance tendencies for more complicated inputs.

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We consider two types of GMFs: the L-filter [or Order Statistic Filter (**1**)] and the M-filter (**3**). For the L-filter, the input value at a point is replaced by a linear combination of the ordered values from a neighborhood or window about the point. The output y_k of an L-filter with window size $W = 2N+1$ at time index k for an input sequence $\{x_i\}$ is given by

$$y_k = \sum_{j=1}^{W} A_j x_{(j)}^k, \tag{1}$$

where $x_{(j)}^k$ is the jth smallest sample among the W samples inside the window centered at k, and where $\{A_j\}_{j=1}^{W}$ is a set of constant weights with

$$\sum_{j=1}^{W} A_j = 1.$$

A particularly simple structure for an L-filter is obtained as follows: Define a positive integer T in terms of a parameter α by $T = [\alpha W]$, where $0 \leqslant \alpha \leqslant 0.5$ and $[x]$ is the largest integer less than or equal to x. The output y_k for the α-Trimmed Mean (α-TM) filter is given by

$$y_k = \sum_{j=T+1}^{W-T} \frac{1}{W-2T} x_{(j)}^k. \tag{2}$$

When $\alpha = 0$ or $\alpha = 1/2$ this L-filter reduces to the usual mean or median filter, respectively (**2**).

The ouput y_k of an M-filter is defined as a solution to the equation

$$\sum_{i=k-N}^{k+N} \psi(x_i - y_k) = 0, \tag{3}$$

where ψ is some odd, continuous and sign-preserving function, so that $\psi(x)$ is positive (negative) whenever x is positive (negative). When ψ is the linear function defined by $\psi(x) = ax$, for any positive constant a, the M-filter reduces to a running mean filter, while it approaches the median filter as ψ approaches the hard limiter (signum) function. A particularly attractive M-filter, the Standard Type M (STM) filter, is obtained with

$$\psi(x) = \begin{cases} 1, & x > p \\ x/p, & |x| \leqslant p \\ -1, & x < -p. \end{cases} \tag{4}$$

The STM filter differs from, but is similar to the Winsorized Mean filter.

We characterize α-TM and STM filter performance by examining the output spectral densities for two cases, the white noise input case and a simple first-order Markov input case, in Sections II and III, respectively. The α-TM filter results are obtained from more general L-filter results. The white noise input results prove useful for noise suppression characterization. However, the nonlinearity of the filters precludes direct extension of the white noise results to general frequency response

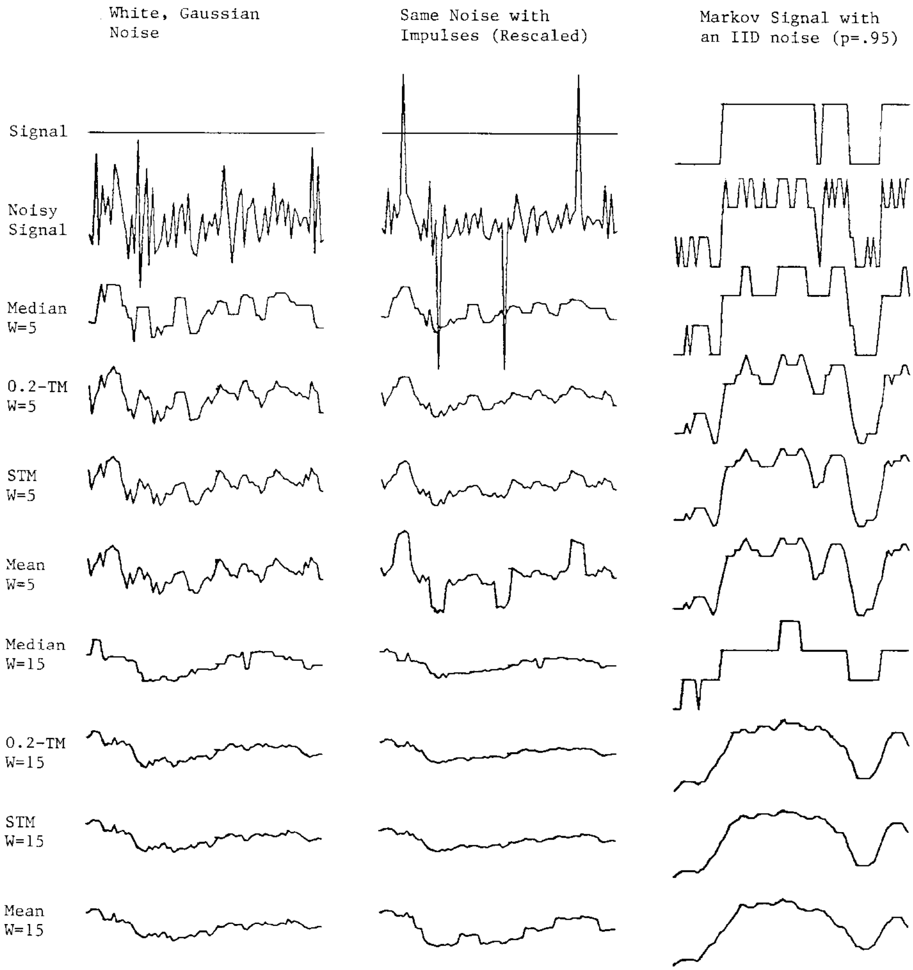

FIG. 1. Outputs from Generalized Median Filters for two window sizes ($W = 5$ and 15) processing Gaussian noise, impulsive white noise and a simple Markov signal with a simple white noise.

interpretations for performance characterization. The first-order Markov input case permits statistical examination of GMF action on edge bearing signals. Extending these results with a simple noise model allows some characterization of noise suppression for the signal plus noise case.

Some sample filter inputs and outputs are given in Fig. 1 to illustrate the class of inputs under consideration and demonstrate various filters in action. Three inputs: white Gaussian noise, white Gaussian noise with impulses, and a noisy Markov signal are filtered by Median, α-TM, STM, and Mean filters with window sizes 5 and 15. These sample signals will be seen to exhibit the filter performance characteristics to be established by the spectral analysis results.

II. Spectral Densities for White Noise Inputs

This section investigates the output spectra of the α-TM and STM filters for a constant input signal corrupted by white noise. The input sample at time k is represented as

$$X_k = S + N_k, \tag{5}$$

where S is a constant signal and N_k is zero-mean, additive white noise. The univariate distribution function of the input X_k is represented as F_x. We derive general formulas for the output correlation functions, which are used to determine output correlations numerically for a Gaussian input distribution and various filter parameters. Then the output spectral properties are discussed.

In determining the correlation function of output samples Y_m and Y_n, we are only concerned with the case $|m-n| < W$, because otherwise, for window size W, Y_m and Y_n are independent. For $|m-n| < W$, the number of input samples, I, contributing to both Y_m and Y_n is $W-|m-n|$. These input samples fall in the overlap between the windows at time positions m and n. Since the inputs are stationary, the correlation function of the output is a function of the integer τ,

$$R(\tau) = E[Y_m Y_{m+\tau}]. \tag{6}$$

Note that for Y_m and $Y_{m+\tau}$ the quantity I is $W-\tau$. The normalized power spectral density is then

$$\Phi(\omega) = \sum_{\tau=-\infty}^{\infty} \frac{R(\tau)}{R(0)} \mathrm{e}^{-j\tau\omega}. \tag{7}$$

Before proceeding, let us define for some m and n, assuming $m \leqslant n$, the sets [as in (7)]

$$M_\mu \triangleq \{\text{samples of the input sequence generating } y_\mu \text{ that are not in the overlap}\} \tag{8}$$

$$Q_{mn} \triangleq \{\text{samples in the overlap}\} \tag{9}$$

where μ is m or n. The sets M_m and M_n each have $W-I$ elements, while Q_{mn} has I elements.

The L-filter output correlation function is readily derived from the joint distribution function of $X^m_{(r)}$ and $X^n_{(s)}$, where $n \triangleq m+\tau$. For each m and n, and hence I, let $B_{i,I}(u)$ be the subset of all sequences for which I samples of the overlap contain exactly i samples which are not larger than u. The joint distribution $P(X^m_{(r)} \leqslant u_m, X^n_{(s)} \leqslant u_n)$ can be written by conditioning on $B_{i,I}(u_n)$ as

$$\sum_{i=0}^{I} P[X^m_{(r)} \leqslant u_m, X^n_{(s)} \leqslant u_n \mid B_{i,I}(u_n)] \tag{10}$$

or

$$\sum_{i=0}^{I} P_1 \cdot P_2 \tag{11}$$

where

$$P_1 \triangleq P\ [\text{at least } r \text{ samples in } M_m U Q_{mn} \text{ are less than or equal to } u_m \mid B_{i,I}(u_n)]$$

and

$$P_2 \triangleq P\ (\text{at least } s-i \text{ samples in } M_n \text{ are less than or equal to } u_n).$$

The probabilities P_1 and P_2 can be written as

$$P_1 = \begin{cases} \sum_{j=r}^{W-I+i} \sum_{t=0}^{i} \binom{W-I}{j-t}^* [P(X \leqslant u_m)]^{j-t}[1-P(X \leqslant u_m)]^{W-I-(j-t)} \\ \times \binom{i}{t}^* [P(X \leqslant u_m \mid X \leqslant u_n)]^t[1-P(X \leqslant u_m \mid X \leqslant u_n)]^{i-t}, & u_m \leqslant u_n \\ 1, & u_m > u_n \text{ and } r \leqslant i \\ \sum_{j=r-i}^{W} \sum_{t=0}^{I-i} \binom{W-I}{j-t}^* [P(X \leqslant u_m)]^{j-t}[1-P(X \leqslant u_m)]^{W-I} \\ \times \binom{I-i}{t}^* [P(X \leqslant u_m \mid X > u_n)]^t[1-P(X \leqslant u_m \mid X > u_n)]^{I-i-t}, & u_m > u_n \text{ and } r > i, \end{cases} \tag{12}$$

and

$$P_2 = \begin{cases} 1, & s \leqslant i \\ \sum_{j=s-i}^{W-I} \binom{W-I}{j}^* [P(X \leqslant u_n)]^j[1-P(X \leqslant u_n)]^{W-I-j}, & s > i, \end{cases} \tag{13}$$

where

$$\binom{a}{b}^* = \begin{cases} \dfrac{a!}{b!(a-b)!}, & a \geqslant b \geqslant 0 \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

These results allow the calculations of the α-TM output correlations. The numerical values of the output correlations for the α-TM filter with window size 5 are given in Table 1. These results are obtained through numerical integration using the above formulas with the input as zero-mean white Gaussian noise with unit variance. Since $R(0)$ is the output variance and the outputs are unbiased, the $R(0)$ results give the output residual mean square error; Table I confirms that the moving mean is optimal for white Gaussian noise suppression, though this is not true for impulsive noise distributions. The normalized power spectral densities are plotted

TABLE I
Output correlations of α-TM filters with window sizes 3 and 5 driven by Gaussian white noise with unit variance

	$W = 3$		$W = 5$		
τ	Median	Running mean	Median	α-TM, $T = 1$	Running mean
0	0.449	0.333	0.287	0.233	0.200
1	0.248	0.222	0.200	0.177	0.160
2	0.118	0.111	0.142	0.129	0.120
3			0.091	0.084	0.080
4			0.044	0.042	0.040

in Fig. 2. All of the normalized power spectral densities show a low-pass characteristic. We can see that the normalized power spectral density of the α-TM filter with $T = 1$ is in between those of the mean and the median filters. The mean (median) filter has the best (worst) low-pass characteristic but has the worst (best) high-pass characteristic. These results partially explain the behaviors of the median and mean filters in suppression of noise and preservation of edges. The α-TM filters can be designed to give better low-pass action than the median filter and better high-pass (edge preservation) characteristics than the mean filter.

We now derive the bivariate distribution function of STM filtered white noise sequences from which the output correlations and power spectral densities are

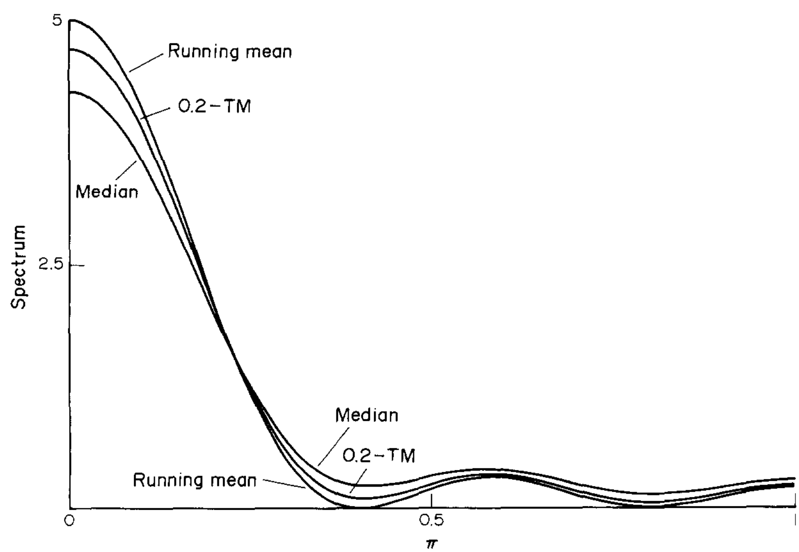

FIG. 2. Output power spectral densities of α-TM filters with window size 5 driven by white Gaussian noise.

readily derived. Let us define the following open and closed intervals which are subsets of the real line, where $0 < p < \infty$ and (i) and (ii) are the conditions $|u_m - u_n| > 2p$ and $|u_m - u_n| \leqslant 2p$, respectively:

$$I_1 \triangleq (-\infty, \min(u_m - p, u_n - p))$$

$$I_2 \triangleq \begin{cases} [\min(u_m - p, u_n - p), \min(u_m + p, u_n + p)], & \text{when (i)} \\ [\min(u_m - p, u_n - p), \max(u_m - p, u_n - p)], & \text{when (ii)} \end{cases}$$

$$I_3 \triangleq \begin{cases} [\min(u_m + p, u_n + p), \max(u_m - p, u_n - p)], & \text{when (i)} \\ [\max(u_m - p, u_n - p), \min(u_m + p, u_n + p)], & \text{when (ii)} \end{cases}$$

$$I_4 \triangleq \begin{cases} [\max(u_m - p, u_n - p), \max(u_m + p, u_n + p)], & \text{when (i)} \\ [\min(u_m + p, u_n + p), \max(u_m + p, u_n + p)], & \text{when (ii)} \end{cases}$$

$$I_5 \triangleq (\max(u_m + p, u_n + p), \infty). \tag{15}$$

Let n_j, $j = 1, 2, 3, 4, 5$ be non-negative integers with

$$0 \leqslant n_j \leqslant I \quad \text{and} \quad \sum_{j=1}^{5} n_j = I.$$

For given m and n we define also the following sets E_j of input random variables for $j = 1, 2, 3, 4, 5$:

$$E_j \triangleq \left\{ X_k \mid X_k \in Q_{mn},\ n - N + \sum_{h=1}^{j-1} n_h \leqslant k \leqslant n - N + \sum_{h=1}^{j} n_h - 1 \right\} \tag{16}$$

where

$$\sum_{h=1}^{j-1} n_h \triangleq 0 \quad \text{for } j = 1.$$

Now, the bivariate distribution function is given by

$$P(Y_m \leqslant u_m, Y_n \leqslant u_n) = P\left(\sum_{M_m U Q_{mn}} \psi(x_k - u_m) \leqslant 0, \sum_{M_n U Q_{mn}} \psi(x_k - u_n) \leqslant 0 \right)$$

$$= \hat{\sum} \hat{C} P\left(\sum_{M_m U Q_{mn}} \psi(x_k - u_m) \leqslant 0, \sum_{M_n U Q_{mn}} \psi(x_k - u_n) \leqslant 0 \mid \right.$$

$$\left. \text{all } x_k\text{'s in } E_j \text{ lie in } I_j \text{ for } j = 1, 2, \ldots, 5 \right)$$

$$\times \prod_{j=1}^{5} P(\text{all } x_k\text{'s in } E_j \text{ lie in } I_j) \tag{17}$$

where

$$\hat{\sum} \triangleq \sum_{n_1=0}^{I} \sum_{n_2=0}^{I_2} \sum_{n_3=0}^{I_3} \sum_{n_4=0}^{I_4}, \tag{18}$$

$$\hat{C} \triangleq \binom{I}{n_1}\binom{I_2}{n_2}\binom{I_3}{n_3}\binom{I_4}{n_4}, \tag{19}$$

and

$$I_h \triangleq I - \sum_{j=1}^{h-1} n_j \quad \text{for } h = 2, 3, 4. \tag{20}$$

In order to determine explicitly the bivariate distribution four separate cases need examination. The two cases for $u_n < u_m$ are determined by interchanging u_n and u_m and using the $u_n \geqslant u_m$ results. The latter is considered as two sub-cases $u_n - u_m > 2p$ and $u_n - u_m \leqslant 2p$. The first sub-case is evaluated below and the second sub-case is considered similarly in (**12**).

For given m, n and k_m, k_n such that $0 \leqslant k_m,\ k_n \leqslant W-I$ define the sets $A_m \triangleq \{x_k | x_k \in M_m,\ m-N \leqslant k \leqslant m-N+k_m-1\}$ and $A_n \triangleq \{x_k | x_k \in M_n,\ m+N+1 \leqslant k \leqslant m+N+k_n\}$. Let

$$P_j^\mu[c, (a,b), (A_\mu, E_j)] \triangleq \begin{cases} P\left[\sum_{A_\mu U E_j} (x_k - u_\mu) \leqslant c,\ |x_k - u_\mu| \leqslant p, \quad \forall x_k \in A_\mu, \right. \\ \left. \qquad a \leqslant x_k - u_\mu \leqslant b,\ \forall x_k \in E_j \right], & A_\mu U E_j \neq \phi \\ 0, & \text{otherwise} \end{cases} \tag{21}$$

where a, b and c are constants with $a < b$, μ is m or n and j is 2 or 4. Then from (17), using (16), we obtain

$$\begin{aligned} P(Y_m \leqslant u_m, Y_n \leqslant u_n) &= \hat{\sum}\hat{C}P\left(\sum_{M_\mu U E_2} \psi(x_k - u_m) \leqslant c_m,\ |x_k - u_m| \leqslant p,\ \forall x_k \in E_2\right) \\ &\times P\left(\sum_{M_n U E_4} \psi(x_k - u_n) \leqslant c_n,\ |x_k - u_n| \leqslant p,\ \forall x_k \in E_4\right) \\ &\times [F_x(u_m - p)]^{n_1}[F_x(u_n - p) - F_x(u_n + p) - F_x(u_m + p)]^{n_3}[1 - F_x(u_n + p)]^{n_5} \end{aligned} \tag{22}$$

where $c_m = (n_1 - n_3 - n_4 - n_5)p$ and $c_n = (n_1 + n_2 + n_3 - n_5)p$. Now

$$\begin{aligned} P\left(\sum_{M_\mu U E_j} \psi(x_k - u_\mu) \leqslant c_\mu,\ |x_k - u_\mu| \leqslant p\ \forall x_k \in E_j\right) &= \sum_{i_\mu=0}^{\hat{I}_\mu} \binom{W-I}{i_\mu} \\ \times [F_x(u_\mu - p)]^{W-I-i_\mu}[1 - F_x(u_\mu + p)]^{i_\mu} &+ \sum_{k_\mu=0}^{W-I} \sum_{i_\mu=0}^{W-I-k_\mu} \binom{W-I}{k_\mu}\binom{W-I-k_\mu}{i_\mu} \\ \times P_j^\mu[c, (-p, p), (A_\mu, E_j)][F_x(u_\mu - p)]^{W_\mu}&[1 - F_x(u_\mu + p)]^{i_\mu} \end{aligned} \tag{23}$$

TABLE II
Output correlations for STM filters with window sizes 3 and 5 driven by Gaussian white noise with unit variance

	$W = 3$				$W = 5$			
τ	Median	STM $p = 1$	STM $p = 2$	Running mean	Median	STM $p = 1$	STM $p = 2$	Running mean
0	0.449	0.363	0.335	0.333	0.287	0.229	0.205	0.200
1	0.248	0.230	0.223	0.222	0.200	0.173	0.161	0.160
2	0.118	0.113	0.112	0.111	0.142	0.127	0.121	0.120
3					0.091	0.084	0.081	0.080
4					0.044	0.041	0.040	0.040

where $\mu = m$ or n, $u = 2$ or 4, $W_\mu = W-I-k_\mu-i_\mu$,

$$\hat{I}_\mu = \min\left[W-I, \frac{1}{2}\left(\frac{c_\mu}{p}+W-I\right)\right]$$

and $c = c_\mu+(W_\mu-i_\mu)p$. The first term in the right hand side of (13) corresponds to the case when $A_\mu UE_j = \phi$. Substituting (23) into (22) yields the bivariate distribution from which the output correlations are readily derived.

The numerical results for the STM filter output correlation function $R(\tau)$ for window sizes 3 and 5 and different p values, when the input is zero-mean white Gaussian noise with unit variance, are presented in Table II. Normalized power

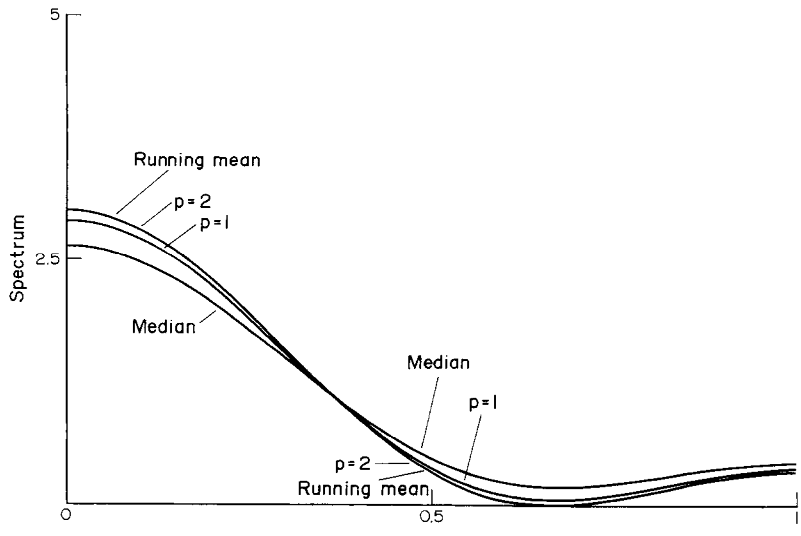

FIG. 3. Output power spectral densities of STM filters with window size 3.

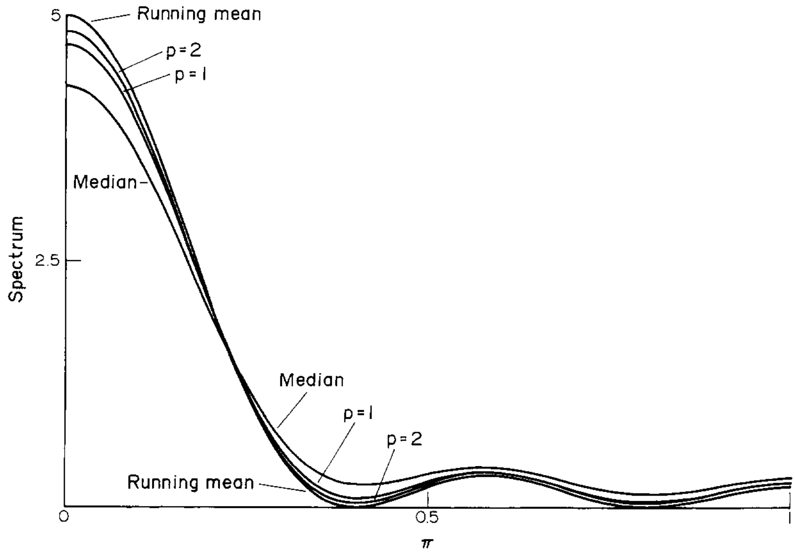

FIG. 4. Output power spectral densities of STM filters with window size 5.

spectra for window sizes 3 and 5 are shown in Figs 3 and 4, respectively. As in α-TM filtering all the normalized power spectral densities show a low-pass characteristic; the filters with window size 5 show a narrower passband than the filters with window size 3. The spectra show that the STM filters compromise between mean and median-like characteristics enabling their design for better non-impulsive noise suppression than the median filter and better high-pass characteristics than the running mean filter.

III. Spectral Densities for a Simple, First Order Markov Input

Thus far we have considered the performance of two GMFs for white noise input signals. The inherent nonlinear character of these GMFs precludes examination of more complicated signals by superposition of simple signal results. While the white noise signals provide great insight into filter characterization and filter design, they are not representative of many useful signals. Kuhlmann and Wise (**13**) have found formulas for the output bivariate probability distribution for k-state Markov chain input signals to the median filter. While this model allows realistic signals, actual

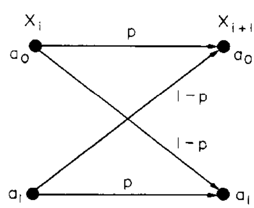

FIG. 5. State Transition Diagram for Binary Symmetric Signal Model.

computation of results can be extremely difficult for k-state Markov chains for even reasonable window sizes. The GMFs introduce additional complexity to the formulation. Hence, to achieve practical results, we primarily restrict ourselves to a two-state first-order Markov signal with equiprobable digits, digit transition probabilities $P(X_i = X_{i-1}) = p$ and $P(X_i \neq X_{i=1}) = 1-p$, where $X_i \in \{a_0, a_1\}$, the set of signal values. This might be called a binary symmetric signal (BSS) model as shown in Fig. 5. For the BSS input class an L- or M-filter output at time i may be computed from knowledge of N_i, the number of a_1's within the filter window centred at the ith input value. This is readily seen by noting that for L- and M-filters, the output is independent of the position of a particular signal value within the window. Rather, dependence is on the set of sample values within the window. Thus, the filter output Y_i is given by a function purely of N_i, $Y_i \triangleq g(N_i)$. As an example, let $W = 5$, $a_1 = 1$, $a_0 = 0$ and the filter be a 1/5-Trimmed Mean. Then $g(5) = 1$, $g(4) = 1$, $g(3) = 2/3$, $g(2) = 1/3$, $g(1) = 0$ and $(0) = 0$.

Since the inputs are not necessarily independent, N_i is not binomially distributed. Rather, the different permutations of n_i values a_1 and $(W-n_i)$ values a_0 will occur with differing probabilities. The straightforward approach to examining the probability of an output y_i is to enumerate all possible sequences yielding output y_i and sum their probabilities. This is somewhat cumbersome for larger window sizes. We will simplify the analysis by considering equivalence classes of sequences with n_i values a_1 where $g(n_i) = y_i$ and by noting that the probability of a BSS sequence occurring is determined by the number of transitions t between the two signal states. Following our previous example, the sequence 01101 occurs with probability $\frac{1}{2}(1-p)(p)(1-p)(1-p) = \frac{1}{2}(1-p)^3(p)^1 = \frac{1}{2}p^{W-t-1}(1-p)^t$ since $t = 3$.

Let $P_w(n, t, x)$ be the probability of the window containing n values a_1 and t transitions from a_1 to a_0 or a_0 to a_1 and a value x at one end of the length W window. Then

$$P_w(n, t, x) = C_w(n, t, x)\tfrac{1}{2}p^{W-t-1}(1-p)^t, \tag{24}$$

where $C_w(n, t, x)$ is the number of distinct W-length sequences with n values a_1, t transitions and end value x. The latter may be conveniently computed recursively by considering a W-length sequence as the concatenation of $(W-1)$-length and 1-length sequences as,

$$C_w(n,t,a_1) = C_{w-1}(n-1,t,a_1)C_1(1,0,a_1)+C_{w-1}(n-1,t-1,a_0)C_1(1,0,a_1) \tag{25}$$

$$C_w(n,t,a_0) = C_{w-1}(n,t-1,a_1)C_1(0,0,a_0)+C_{w-1}(n,t,a_0)C_1(0,0,a_0). \tag{26}$$

Clearly,

$$C_w(0, 0, 0) = C_w(W, 0, 1) = 1 \tag{27}$$

$$C_1(n, t, 0) = \begin{cases} 1, & n = t = 0 \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

and

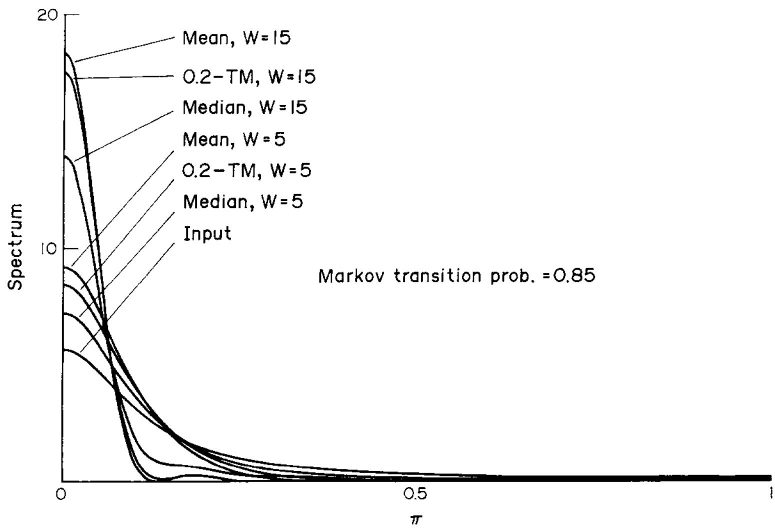

FIG. 6. The input and output spectral densities for the BSS model with transition probability 0.85 for several GMFs with window sizes 5 and 15.

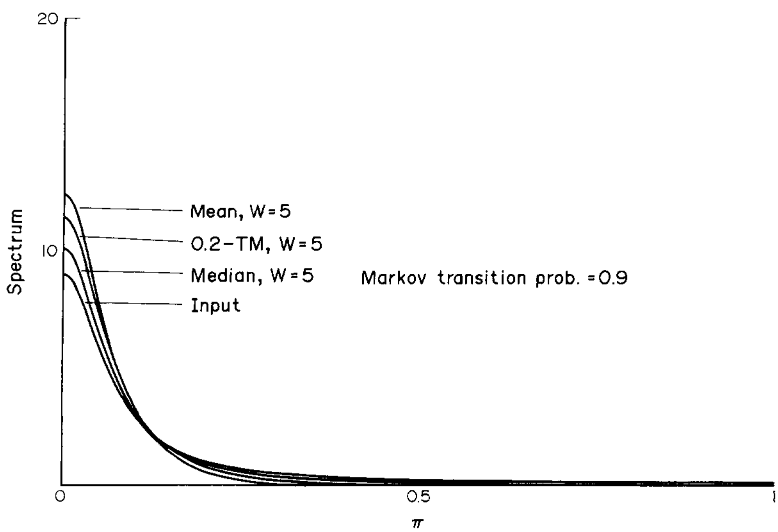

FIG. 7. The input and output spectral densities for the BSS model with transition probability 0.9 for several GMFs with window size 5.

$$C_1(n, t, 1) = \begin{cases} 1, & n = 1 \text{ and } t = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

The number of transitions is constrained by $0 \leqslant t \leqslant \min(2n, 2W-2n)$. Values of $C_w(n, t, x)$ are considered to be zero for $n < 0$, $t < 0$ or $t > \min(2n, 2W-2n)$. Substituting into (25) and (26), yields

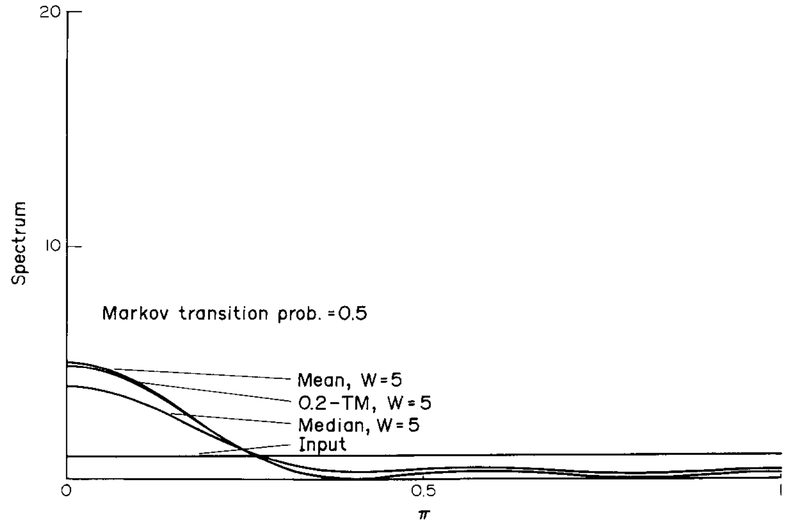

FIG. 8. The input and output spectral densities for the BSS model with transition probability 0.5 for several GMFs with window size 5.

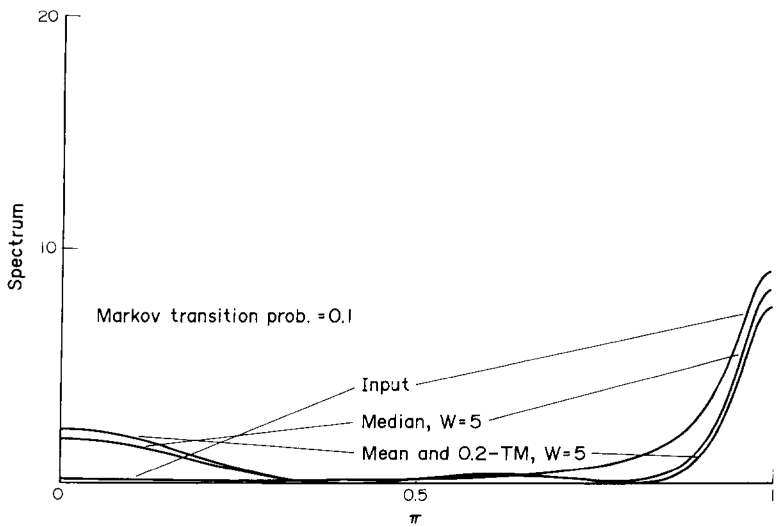

FIG. 9. The input and output spectral densities for the BSS model with transition probability 0.1 for several GMFs with window size 5.

$$C_w(n, t, a_1) = C_{w-1}(n-1, t, a_1) + C_{w-1}(n-1, t-1, a_0) \tag{30}$$

$$C_w(n, t, a_0) = C_{w-1}(n, t-1, a_1) + C_{w-1}(n, t, a_0). \tag{31}$$

The bivariate distribution of Y_i and Y_0 is calculated by summing over the equivalence class of sequences with $g(N_i) = Y_i$ and $g(N_0) = Y_0$, that is

$$P_{Y_i,Y_0}(y_i, y_0) = \sum_{\substack{\forall n_i | g(n_i) = y_i \\ \forall n_0 | g(n_0) = y_0}} P_{N_i,N_0}(n_i, n_0). \tag{32}$$

The bivariate distribution, $P_{N_i,N_0}(n_i, n_0)$, of N_i and N_0 can be evaluated by considering the overlapping and separated window cases. For $i \geqslant W$, the two windows are fully separated, but the outputs are not necessarily independent due to the first-order Markov input usually exhibiting dependence ($p \neq 0.5$). By considering the sequences inside and between the windows, conditioning on the end digits of the sequences, invoking the Markov property and exploiting the binary digit restriction, the bivariate distribution can be derived as

$$P_{N_i,N_0}(n_i, n_0) = \frac{1}{2} \sum_{x_N \in \{a_0, a_1\}} \left[\sum_{x_{i-N} \in \{a_0, a_1\}} P_w(n_i \mid x_{i-N}) P(x_N \mid x_{i-N}) \right] P_w(n_0 \mid x_N), \tag{33}$$

where

$$P_w(n \mid x) = \sum_{t=0}^{\min(2n, 2W-2n)} C_w(n, t, x) p^{W-t-1} (1-p)^t. \tag{34}$$

By Markov chain analysis for the BSS model, we have

$$P(x_N \mid x_{i-N}) = \begin{cases} \frac{1}{2}[1 + (2p-1)]^{i-2N}, & x_N = x_{i-N} \\ \frac{1}{2}[1 - (2p-1)]^{i-2N}, & x_N \neq x_{i-N}. \end{cases} \tag{35}$$

Similarly, results for the overlapping case, $0 \leqslant i < W$ can be derived: i.e.

$$P_{N_i,N_0}(n_i, n_0) = \sum_{m_1=0}^{W} \sum_{x_{i-N-1}} \sum_{x_{i-N}} \sum_{x_N} \sum_{x_{N+1}} P_i(n_i - m_1 \mid x_{N+1})$$

$$\times P_i(n_0 - m_1 \mid x_{i-N-1}) p^{A+B} (1-p)^{2-(A+B)} \sum_{\gamma=0}^{G} P_{w-i}(m_1, 2\gamma + D, x_N), \tag{36}$$

where

$$A = \begin{cases} 1, & x_{N+1} = x_N \\ 0, & \text{otherwise,} \end{cases} \tag{37}$$

$$B = \begin{cases} 1, & x_{i-N-1} = x_{i-N} \\ 0, & \text{otherwise,} \end{cases} \tag{38}$$

$$D = \begin{cases} 1, & x_N \neq x_{i-N} \\ 0, & \text{otherwise,} \end{cases} \tag{39}$$

and

$$G = \left[\frac{W - i - D - 1}{2} \right]. \tag{40}$$

Results for a BSS plus noise input model have been readily derived by extending the above BSS results, but are omitted here for brevity. Using the above results, comparisons of output spectra are now made.

Figures 6–9 show the input and output power spectral densities for several Markov transition probabilities for the two-state BSS model. Output results are given for the Median, 0.2-Trimmed Mean, and Mean filters. For this restricted BSS input, an STM can be designed to perform as either a Median or a Mean filter. The results indicate that all the filters have a lowpass characteristic with more pronounced lowpass action the larger the filter window size relative to expected BSS run length $(1/1-p)$. The α-TM filter compromises performance between that of the Median and the Mean filter for the BSS input.

IV. Conclusions

We have given the results to calculate the output correlations and power spectral densities for several Generalized Median Filters including the Median, Mean-, L-, α-Trimmed Mean, M-, and Standard Type M-filters for white noise and first-order binary Markov sequence inputs. Generally, the filters act as smoothers, exhibiting a lowpass action. The GMFs effectively compromise edge bearing signal retention with noise suppression.

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