Diversity–Multiplexing Tradeoff and Outage Performance for Rician MIMO Channels

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***Abstract*—In this paper, we analyze the diversity–multiplexing tradeoff (DMT), originally introduced by Zheng and Tse, and outage performance for Rician multiple-input–multiple-output (MIMO) channels. The DMT characteristics of Rayleigh and Rician channels are shown to be identical. In a high signal-to-noise ratio (SNR) regime, the log–log plot of outage probability versus SNR curve for a Rician channel is a shifted version of that for the corresponding Rayleigh channel. The SNR gap between the outage curves of the Rayleigh and Rician channels is derived. The DMT and outage performance are also analyzed for Rician multiple-input–single-output (MISO)/single-input–multiple-output (SIMO) channels over a finite SNR regime. A closed-form expression for the outage probability is derived and the finite SNR DMT characteristic is analyzed. It is observed that the maximum diversity gain can be achieved at some finite SNR–the maximum gain tends to increase linearly with the Rician factor. The finite SNR diversity gain is shown to be a linear function of the finite SNR multiplexing gain. The consistency between the DMTs for finite and infinite SNRs is also shown.**

***Index Terms*—Diversity–multiplexing tradeoff (DMT), finite signal-to-noise ratio (SNR), maximum diversity gain, multiple-input–multiple-output (MIMO), outage, Rician.**

I. Introduction

MULTIPLE-INPUT–MULTIPLE-OUTPUT (MIMO) antenna systems can offer a dramatic increase in transmission rates by spatial multiplexing [1]–[3], and they can significantly improve link reliability by transmit diversity such as space–time coding [4], [5]. However, their design requires some caution because there is a tradeoff between diversity and multiplexing gains and maximizing one type of gain does not necessarily maximize the other. This type of tradeoff was first discussed by Tarokh *et al.* [4], where an upper bound on the symbol rate is derived as a function of the transmit diversity gain and signal constellation size. Tarokh *et al.* define the symbol rate as the number of transmit symbols per channel use (pcu) and the transmit diversity gain in terms of the pairwise error probability.

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Based on this result, various space–time codes have been developed, including the full-rate and full-diversity codes [6]–[9], linear dispersion codes [10], [11], and space–time codes with optimal rate-diversity tradeoff [12]–[14].

An alternative approach to characterizing the tradeoff has been introduced by Zheng and Tse [15]. In the high signal-to-noise ratio (SNR) regime, they define the diversity gain in terms of codeword error probability and the multiplexing gain as the normalized rate parameter r, where the limiting transmission rate is given by $r\log\text{SNR}$ bits pcu (it scales with SNR). Then, the maximum-achievable diversity gain is obtained for each r under the following assumptions: the channel is independent and identically distributed (i.i.d.) Rayleigh and quasi-static, the SNR approaches infinity, and perfect channel state information (CSI) is available at the receiver but there is no CSI at the transmitter. In their derivation, outage analysis plays an important role due to the fact that outage and codeword error probabilities have the same SNR exponent, which represents the diversity gain. This work has stimulated a number of research efforts to extend the optimal diversity–multiplexing tradeoff (DMT) for other practical channel models [16]–[19] and for cases with some CSI at the transmitter [20]–[22]. Asymptotic outage performance [23] and the DMT for finite SNR [24], [25] have also been characterized. In addition, there have been many works that propose practical techniques approaching the optimal tradeoff [26]–[32].

A recent trend in DMT research is a compound channel approach [27], [28], [33], where a slowly varying fading channel not in outage is viewed as a set of different channel realizations for which optimal codes are designed. Especially, approximately universal codes [27], [28] have been proposed based on compound channel approaches. They showed that a code satisfying a certain criterion achieves the optimal DMT for MIMO channels with arbitrary fading statistics. Such a design methodology is also practically important since it is robust to channel modeling errors. This approach can be applied to any fading distributions including Rayleigh and Rician, thus providing a general framework for studying DMT.

Such general approaches are ideal for studying DMT and outage behaviors at asymptotically high SNRs. However, it would also be beneficial to study outage behaviors at finite SNRs that are practical operating regimes. Furthermore, such outage behaviors at finite SNRs can be quite different for different fading statistics, which may not show up in DMT analysis alone.

In this paper, we study the DMT and outage behaviors for Rician MIMO channels. Assuming no CSI at the transmitter for slowly varying Rayleigh fading channels can be pessimistic in practical systems because slow fading means there is enough time for channel state feedback and thus assuming partial CSI would be more realistic if there existed a feedback path. Such CSI would not be perfect due to time-varying fading, channel estimation errors, and errors in the feedback channel. Rician fading can be a good model for such partial CSI scenarios, i.e., the mean part representing the known part and the Rayleigh part representing the uncertainty of the CSI knowledge, and therefore can be important in practice.

We first show that the DMT characteristics of Rayleigh and Rician channels are identical, which is rather obvious. This fact indicates that for high SNR, the log–log plot of outage probability versus the SNR curve for a Rician channel is a shifted version of that for the corresponding Rayleigh channel. We analyze the SNR difference between the outage curves for Rayleigh and Rician channels. Furthermore, the finite SNR DMT is characterized for Rician multiple-input–single-output (MISO)/single-input–multiple-output (SIMO) channels through some modification of the results in [24] and [25]. Specifically, an alternative definition for finite SNR multiplexing gain is introduced, while adopting the same definition as in [24], [25] for diversity gains, and a closed-form expression for the outage probability is derived in terms of the generalized Marcum Q-function [34]. Using the outage expression for MISO/SIMO channels, closed-form expressions for the maximum diversity gain and the corresponding finite SNR, at which the maximum is achieved, are derived. The finite SNR diversity gain for MISO/SIMO channels is represented as a linear function of the finite SNR multiplexing gain.

The rest of this paper is organized as follows. Section II describes the model for a Rician MIMO channel and Section III characterizes the DMT and outage performance for Rician channels. The maximum diversity gain and DMT for finite SNR are derived in Section IV for the case of Rician MISO/SIMO channels. Finally, Section V summarizes this paper with some concluding remarks.

Throughout this paper the superscript † denotes the conjugate transpose of a matrix, $\mathbf{I}_n$ is the identity matrix of size $n \times n$, and $\mathbb{C}$ is the field of complex numbers. $E[\cdot]$, $\mathrm{tr}(\cdot)$, and $\det(\cdot)$ are the expectation, the trace, and the determinant, respectively. Unless otherwise stated, all logarithms are assumed to be to the base 2.

II. Channel and System Model

Consider a MIMO system with n_T transmit antennas and n_R receive antennas over a flat fading channel. Let $\mathbf{x} \in \mathbb{C}^{n_T}$ and $\mathbf{y} \in \mathbb{C}^{n_R}$ denote the transmitted and received vectors, respectively, at a certain time. The baseband equivalent signal model of the MIMO system is given by

$$\mathbf{y} = \sqrt{\frac{\rho}{n_T}}\mathbf{H}\mathbf{x} + \mathbf{n} \tag{1}$$

where ρ is the average SNR at each receive antenna, $\mathbf{H} \in \mathbb{C}^{n_R \times n_T}$ is the channel matrix whose element h_{ij} represents the complex channel gain between the jth transmit antenna and the ith receive antenna, and $\mathbf{n} \in \mathbb{C}^{n_R}$ is an additive white Gaussian noise (AWGN) vector. For notational convenience, we define

$$m = \min\{n_T, n_R\} \quad \text{and} \quad n = \max\{n_T, n_R\}. \tag{2}$$

When the channel is Rician, $\mathbf{H}$ can be expressed as

$$\mathbf{H} = \sqrt{\frac{K}{K+1}}\bar{\mathbf{H}} + \sqrt{\frac{1}{K+1}}\mathbf{H}_w \tag{3}$$

where K denotes the Rician factor ($K \geq 0$), $\bar{\mathbf{H}}$ is a deterministic matrix representing line-of-sight (LOS) components and is normalized as $\mathrm{tr}(\bar{\mathbf{H}}\bar{\mathbf{H}}^\dagger) = n_R n_T$, and $\mathbf{H}_w$ consists of zero-mean complex Gaussian random variables with unit variance [35]. Hence, the channel is normalized such that

$$E[\mathrm{tr}(\mathbf{H}\mathbf{H}^\dagger)] = n_R n_T. \tag{4}$$

It is assumed that the channel $\mathbf{H}_w$ is known at the receiver, but not known at the transmitter. Both the transmitter and the receiver know $\bar{\mathbf{H}}$.

III. DMT and Outage Probability for Rician Channels

In this section, the DMT for Rician channels is examined after briefly reviewing the DMT for Rayleigh channels. Then, an asymptotic difference between the outage probabilities of Rayleigh and Rician channels is analyzed.

A. DMT for Rayleigh Channels [15]

Let r and d denote the multiplexing and diversity gains, respectively. Then

$$r = \lim_{\rho\to\infty} \frac{R(\rho)}{\log\rho} \tag{5a}$$

and

$$d = -\lim_{\rho\to\infty} \frac{\log P_e(\rho)}{\log\rho} \tag{5b}$$

where $R(\rho)$ is the target rate for a given SNR ρ and $P_e(\rho)$ is the error probability assuming the maximum-likelihood (ML) decoding.[1] For simplicity, the notation $\doteq$ is used for representing the relation in (5b), specifically

$$P_e(\rho) \doteq \rho^{-d} \tag{6}$$

is identical to (5b), and $\doteq$ is referred to as the exponential equality.

The optimal DMT for MIMO Rayleigh channels is described as follows.

Theorem 1 [15]: When the channel in (1) has Rayleigh distribution and the block length $l \geq n_T + n_R - 1$, the optimal DMT curve is given by the piecewise linear function connecting the points $(k, (n_T - k)(n_R - k))$ for $k = 0, 1, \ldots, \min\{n_T, n_R\}$.

The optimal DMT curve represents the maximum diversity gain for a given multiplexing gain r and is denoted by $d^*(r)$. This theorem has been proved by showing that the outage probability

$$P_{\mathrm{out}}(R, \rho) = \Pr\left\{\log\det\left(\mathbf{I}_{n_R} + \frac{\rho}{n_T}\mathbf{H}\mathbf{H}^\dagger\right) < R\right\} \tag{7}$$

satisfies

$$P_{\mathrm{out}}(R, \rho) \doteq \rho^{-d^*(r)} \tag{8}$$

[1]To simplify notations, $R(\rho)$ will be written as R if dropping ρ does not cause any confusion.

and that the error probability $P_e(\rho)$ of an optimal DMT-achieving scheme also satisfies

$$P_e(\rho) \doteq \rho^{-d^*(r)}. \quad (9)$$

B. DMT for Rician Channels

For a Rician channel with factor K, the outage probability, denoted by $P_{\text{out}}(R,\rho,K)$, is defined by (7) and can be rewritten as

$$P_{\text{out}}(R,\rho,K) = \Pr\left\{\log\left(\prod_{l=1}^{m}\left(1+\frac{\rho}{n_T}\lambda_l\right)\right) < R\right\} \quad (10)$$

where $\{\lambda_l\}$ are the eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ and $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$. The joint probability density function (pdf) of the ordered eigenvalues $\{\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m\}$ is derived as follows.

Lemma 1: Let $\phi_1, \phi_2, \ldots, \phi_m$ denote the m eigenvalues of $\bar{\mathbf{H}}\bar{\mathbf{H}}^\dagger$, where $\bar{\mathbf{H}}$ is the channel mean matrix. Suppose that $\phi_i \neq \phi_j$ for all i and j. Then, the joint pdf of the ordered eigenvalues $\{\lambda_l\}$ is given by

$$\begin{aligned} f(\lambda_1,\ldots,\lambda_m) &= C_1 K^{-\bar{m}}(K+1)^{-\bar{m}+mn}\prod_{k<l}^{m}(\phi_k-\phi_l)^{-1} \\ &\cdot \prod_{k=1}^{m}\lambda_k{}^{n-m}\prod_{k<l}^{m}(\lambda_k-\lambda_l)e^{-K\sum_l \phi_l}e^{-\sum_l (K+1)\lambda_l} \\ &\cdot \det(\{{}_0F_1(n-m+1, K(K+1)\phi_k\lambda_l)\}) \end{aligned} \quad (11)$$

where C_1 is a normalizing constant, $\bar{m} = m(m-1)/2$, $\det(\{a_{ij}\})$ indicates the determinant of a matrix whose (i,j)th entry is a_{ij}, and ${}_0F_1(\cdot,\cdot)$ is a hypergeometric function with scalar arguments given by

$${}_0F_1(n-m+1;x) = (n-m)!x^{-\frac{(n-m)}{2}}I_{n-m}(2\sqrt{x}). \quad (12)$$

Here, $I_{n-m}(\cdot)$ is the modified Bessel function of the first kind [36].

This result can be obtained by modifying the joint pdf $f(\lambda_1,\ldots,\lambda_m)$ in [37] using the hypergeometric function expression in [38]. If the Rician factor K becomes zero, (11) should be the same as the joint pdf of $\{\lambda_l\}$ for MIMO Rayleigh channels in [15]. Equation (11) cannot be evaluated when K is exactly zero, but its limit as $K \to 0$ exists. Next, an asymptotic expression for the determinant of the hypergeometric function is derived.

Lemma 2: For high SNR, $\det(\{{}_0F_1(n-m+1, K(K+1)\phi_k\lambda_l)\})$ in (11) can be represented as

$$\begin{aligned} &\det(\{{}_0F_1(n-m+1,K(K+1)\phi_k\lambda_l)\}) \\ &= \frac{[(n-m)!]^m[K(K+1)]^{\bar{m}}}{\prod_{k=1}^{m-1}k!(n-m+k)!} \\ &\cdot \left[\prod_{k<l}^{m}(\phi_k-\phi_l)(\lambda_k-\lambda_l)\right](1+O(\lambda_m)) \end{aligned} \quad (13)$$

where $f(x) = O(g(x))$ means that positive constants M and m exist such that $f(x) \le Mg(x)$ for all $x > m$.

The proof of this lemma is presented in Appendix I. Note that when setting $K=0$ and using (13) in (11), the joint pdf of $\{\lambda_l\}$ becomes irrelevant of $\{\phi_i\}$ for all i and is thus equivalent to that for Rayleigh channels. It can be easily verified by putting $K=0$ into (14). Using Lemmas 1 and 2, we obtain the desired result.

Theorem 2: The optimal DMT curve for Rician channels is independent of the Rician factor K and is identical to that for Rayleigh channels.

Proof: Using (13) in (11), we get

$$\begin{aligned} f(\lambda_1,\ldots,\lambda_m) &= C_2(K+1)^{mn}\prod_{k=1}^{m}\lambda_k{}^{n-m}\prod_{k<l}^{m}(\lambda_k-\lambda_l)^2 \\ &\cdot e^{-K\sum_l \phi_l}e^{-\sum_l (K+1)\lambda_l}(1+O(\lambda_m)) \end{aligned} \quad (14)$$

where

$$C_2 = \frac{C_1[(n-m)!]^m}{\prod_{k=1}^{m-1}k!(n-m+k)!}. \quad (15)$$

Following the approach in [15], we define

$$\alpha_l = -\frac{\log\lambda_l}{\log\rho}. \quad (16)$$

The joint pdf of $\{\alpha_1,\ldots,\alpha_m\}$ is

$$\begin{aligned} &f(\alpha_1,\ldots,\alpha_m) \\ &= C_2(K+1)^{mn}(\log\rho)^n \\ &\cdot \prod_{k=1}^{m}\rho^{-(n-m+1)\alpha_k}\prod_{k<l}^{m}(\rho^{-\alpha_k}-\rho^{-\alpha_l})^2 \\ &\cdot e^{-K\sum_l \phi_l}e^{-\sum_l (K+1)\rho^{-\alpha_l}}\left(1+O\left(\frac{1}{\rho^{\alpha_m}}\right)\right) \end{aligned} \quad (17)$$

and the outage probability is given by

$$P_{\text{out}}(R,\rho,K) \doteq \int_{\mathcal{A}} f(\alpha_1,\ldots,\alpha_m)d\alpha_1\cdots d\alpha_m \quad (18)$$

where set $\mathcal{A}$ consists of $\{(\alpha_1,\ldots,\alpha_m)\}$ for which the outage occurs. In (17), note that

$$\lim_{\rho\to\infty}\frac{\log\left(C_2(K+1)^{mn}(\log\rho)^n e^{-K\sum_l \phi_l}\right)}{\log\rho} = 0$$

and for any $\alpha_l < 0$, the term $e^{-\sum_l (K+1)\rho^{-\alpha_l}}$ decays exponentially with ρ. Due to these facts, (18) can be rewritten as

$$\begin{aligned} P_{\text{out}}(R,\rho,K) \doteq \int_{\mathcal{A}'} &\prod_{k=1}^{m}\rho^{-(n-m+1)\alpha_k} \\ &\cdot \prod_{k<l}^{m}(\rho^{-\alpha_k}-\rho^{-\alpha_l})^2 d\alpha_1\cdots d\alpha_m \end{aligned} \quad (19)$$

where $\mathcal{A}'$ denotes $\mathcal{A}\cap\mathcal{R}^{m+}$ and $\mathcal{R}^{m+}$ represents the set of m-real vectors whose elements are nonnegative. The outage expression in (19) is identical to that for Rayleigh channels. This completes the proof. □

Theorem 2 indicates that LOS components of Rician channels do not influence the DMT property of the MIMO system. This happens because the multiplexing and diversity gains are defined when SNR approaches infinity—in this case, the tail behaviors of Rician channels follow those of Rayleigh channels.

If the Rician factor K reaches infinity, the Rician channel matrix $\mathbf{H}$ in (3) becomes $\bar{\mathbf{H}}$, which is deterministic. The DMT for this case is described below.

Theorem 3: When the Rician factor K becomes infinite, the optimal DMT curve is given by

$$d^*(r) = \begin{cases} \infty, & \text{if } 0 \le r < \text{rank}(\bar{\mathbf{H}}) \\ 0, & \text{if } \text{rank}(\bar{\mathbf{H}}) < r \le \min\{n_T, n_R\}. \end{cases} \quad (20)$$

Proof: From (10), the outage probability for $\bar{\mathbf{H}}$ can be written as

$$P_{\text{out}}(R,\rho) = \Pr\left\{\log\left(\prod_{l=1}^{m}\left(1+\frac{\rho}{n_T}\phi_l\right)\right) < R\right\} \quad (21)$$

where $\{\phi_l\}$ are the eigenvalues of $\bar{\mathbf{H}}\bar{\mathbf{H}}^\dagger$. From (5a), $2^R \doteq \rho^r$, and (21) is rewritten as

$$\begin{aligned} P_{\text{out}}(R,\rho) &= \Pr\left\{\prod_{l=1}^{\text{rank}(\bar{\mathbf{H}})}\left(1+\frac{\rho}{n_T}\phi_l\right) < 2^R\right\} \\ &\doteq \Pr\left\{\prod_{l=1}^{\text{rank}(\bar{\mathbf{H}})}\left(1+\frac{\rho}{n_T}\phi_l\right) < \rho^r\right\} \\ &\doteq \Pr\left\{\rho^{\text{rank}(\bar{\mathbf{H}})} < \rho^r\right\}. \end{aligned} \quad (22)$$

Hence, $d^*(r)$ from (5b) is expressed as

$$d^*(r) = -\lim_{\rho\to\infty}\frac{\log\Pr\left\{\rho^{\text{rank}(\bar{\mathbf{H}})} < \rho^r\right\}}{\log\rho}. \quad (23)$$

If $0 \le r < \text{rank}(\bar{\mathbf{H}})$, then $\Pr\{\rho^{\text{rank}(\bar{\mathbf{H}})} < \rho^r\} = 0$ and the right-hand side (RHS) of (23) becomes ∞. For $\text{rank}(\bar{\mathbf{H}}) < r \le \min\{n_T, n_R\}$, $\Pr\{\rho^{\text{rank}(\bar{\mathbf{H}})} < \rho^r\} = 1$ and $d^*(r) = 0$. □

Note that for the deterministic channel matrix $\bar{\mathbf{H}}$, we have infinite diversity gain as long as the multiplexing gain $r < \text{rank}(\bar{\mathbf{H}})$.

C. Asymptotic Difference Between Outage Probabilities of Rayleigh and Rician Channels

Theorem 2 indicates that in a high SNR regime, the log–log plot of outage probability versus SNR curve for a Rician channel is a shifted version of that for the corresponding Rayleigh channel. The SNR difference between the outage curves for Rayleigh and Rician channels is called the *asymptotic SNR gap*. To be specific, let

$$P_{\text{out}}(R,\rho_{RI},K) = P_{\text{out}}(R,\rho_{RA},0) = P_o(\rho_{RI},\rho_{RA}) \quad (24)$$

where $P_o(\rho_{RI},\rho_{RA})$ is a desired outage probability for a fixed target rate R. ρ_{RI} and ρ_{RA} are the SNR values that are required for achieving $P_o(\rho_{RI},\rho_{RA})$ when the channels are Rician and Rayleigh channels, respectively. Our objective is to find the difference between $\log\rho_{RA}$ and $\log\rho_{RI}$ in a high SNR regime such that $P_{\text{out}}(R,\rho_{RI},K) = P_{\text{out}}(R,\rho_{RA},0)$ is satisfied. Let

$$\Delta = \frac{\rho_{RA}}{\rho_{RI}}. \quad (25)$$

Then, $\log\Delta = \log\rho_{RA} - \log\rho_{RI}$ is the SNR gap. Its asymptotic properties are derived using the following lemma.

Lemma 3: For high SNR, the outage probability in (10) can be expressed as

$$P_{\text{out}}(R,\rho,K) = C_3 R^m \rho^{-mn}(K+1)^{mn} e^{-K\sum_l \phi_l} \cdot [t(R,\rho,K) + \epsilon(R,\rho,K)] \quad (26a)$$

where

$$C_3 = \frac{C_1(\ln 2)^m n_T^{mn}[(n-m)!]^m}{\prod_{k=1}^{m-1} k!(n-m+k)!}, \quad (26b)$$

$$t(R,\rho,K) = \int_{\mathcal{A}} 2^{R\sum_l \beta_l}\prod_{k=1}^{m}(2^{\beta_k R}-1)^{n-m}\prod_{k<l}^{m}(2^{\beta_k R}-2^{\beta_l R})^2 \cdot e^{-\sum_l \frac{(K+1)n_T(2^{\beta_l R}-1)}{\rho}} d\beta_1\cdots d\beta_m, \quad (26c)$$

$$\epsilon(R,\rho,K) = \int_{\mathcal{A}} 2^{R\sum_l \beta_l}\prod_{k=1}^{m}(2^{\beta_k R}-1)^{n-m}\prod_{k<l}^{m}(2^{\beta_k R}-2^{\beta_l R})^2 \cdot e^{-\sum_l \frac{(K+1)n_T(2^{\beta_l R}-1)}{\rho}} O\left(\frac{1}{\rho}\right) d\beta_1\cdots d\beta_m, \quad (26d)$$

$\beta_l = \log(1+\frac{\rho}{n_T}\lambda_l)/R$, and $\mathcal{A}$ represents the hyperspace of $\{\beta_1,\ldots,\beta_m\}$ satisfying $0 < \beta_1 \le \cdots \le \beta_m$ and $1 - \sum_l \beta_l > 0$.

Proof: From Lemmas 1 and 2, the joint pdf of $\{\beta_1,\ldots,\beta_m\}$ for high SNR is represented as

$$\begin{aligned} &f(\beta_1,\ldots,\beta_m) \\ &= \frac{C_1[(n-m)!]^m}{\prod_{k=1}^{m-1} k!(n-m+k)!}(\ln 2)^m n_T^{mn} R^m \rho^{-m} \\ &\quad\cdot 2^{R\sum_l \beta_l}(K+1)^{mn}\prod_{k<l}^{m}\left(\frac{n_T(2^{\beta_k R}-1)}{\rho}\right)^{n-m} \\ &\quad\cdot\prod_{k<l}^{m}\left(\frac{n_T(2^{\beta_k R}-2^{\beta_l R})}{\rho}\right)^2 e^{-K\sum_l \phi_l} \\ &\quad\cdot e^{-\sum_l \frac{(K+1)n_T(2^{\beta_l R}-1)}{\rho}}\left(1+O\left(\frac{1}{\rho}\right)\right) \\ &= \frac{C_1[(n-m)!]^m}{\prod_{k=1}^{m-1} k!(n-m+k)!}(\ln 2)^m n_T^{mn} R^m \\ &\quad\cdot\rho^{-mn}(K+1)^{mn} e^{-K\sum_l \phi_l} 2^{R\sum_l \beta_l} \\ &\quad\cdot\prod_{k=1}^{m}(2^{\beta_k R}-1)^{n-m}\prod_{k<l}^{m}(2^{\beta_k R}-2^{\beta_l R})^2 \\ &\quad\cdot e^{-\sum_l \frac{(K+1)n_T(2^{\beta_l R}-1)}{\rho}}\left(1+O\left(\frac{1}{\rho}\right)\right) \\ &= C_3 R^m \rho^{-mn}(K+1)^{mn} e^{-K\sum_l \phi_l} 2^{R\sum_l \beta_l} \\ &\quad\cdot\prod_{k=1}^{m}(2^{\beta_k R}-1)^{n-m}\prod_{k<l}^{m}(2^{\beta_k R}-2^{\beta_l R})^2 \\ &\quad\cdot e^{-\sum_l \frac{(K+1)n_T(2^{\beta_l R}-1)}{\rho}}\left(1+O\left(\frac{1}{\rho}\right)\right). \end{aligned} \quad (27)$$

Equation (27) follows the steps similar to those in [23], which are essentially equivalent to those of the original DMT paper [15] except that $\beta_l = \log(1+\frac{\rho}{n_T}\lambda_l)/R$ is used instead of $\alpha_l = -\frac{\log\lambda_l}{\log\rho}$. Because (10) is given by the integral of the joint pdf of $\{\beta_1,\ldots,\beta_m\}$ over the hyperspace $\mathcal{A}$, the outage probability in (26a) is obtained using (27) in

$$P_{\rm out}(R,\rho,K) = \int_{\mathcal{A}} f(\beta_1,\ldots,\beta_m)d\beta_1\cdots d\beta_m. \quad (28)$$

□

Note that for the Rayleigh channel, $K=0$ and the outage probability is given by [37]

$$P_{\rm out}(R,\rho,0) = C_3R^m\rho^{-mn}t(R,\rho,0). \quad (29)$$

Theorem 4: For the $n_T \times n_R$ MIMO Rayleigh and Rician channels, the asymptotic SNR gap is given by

$$\lim_{\rho_{RI}\to\infty}\Delta = \frac{e^K}{K+1} \quad (30)$$

where K is the Rician factor and the limit is taken such that $P_{\rm out}(R,\rho_{RI},K) = P_{\rm out}(R,\rho_{RA},0)$ is satisfied.

Proof: Because the asymptotic diversity gains for Rayleigh and Rician channels are identical, $\log\Delta$ converges to a constant as ρ_{RI} and ρ_{RA} go to infinity. Since $\lim_{\rho\to\infty} e^{-\sum_l \frac{(K+1)n_T(2^{\beta_l R}-1)}{\rho}} = 1$, from (26c), we obtain

$$\lim_{\rho_{RI}\to\infty}\frac{t(R,\rho_{RI},K)}{t(R,\rho_{RA},0)} = 1.$$

In (26d), $\lim_{\rho_{RI}\to\infty}\epsilon(R,\rho_{RI},K) = 0$ and $\lim_{\rho_{RA}\to\infty}\epsilon(R,\rho_{RA},0) = 0$. Hence

$$\begin{aligned}\lim_{\rho_{RI}\to\infty}&\frac{P_{\rm out}(R,\rho_{RI},K)}{P_{\rm out}(R,\rho_{RA},0)}\\ &= \lim_{\rho_{RI}\to\infty}\frac{(K+1)^{mn}e^{-K\sum_l\phi_l}t(R,\rho_{RI},K)}{\Delta^{-mn}t(R,\rho_{RA},0)}\\ &= \lim_{\rho_{RI}\to\infty}\Delta^{mn}(K+1)^{mn}e^{-K\sum_l\phi_l}\\ &= \lim_{\rho_{RI}\to\infty}\left(\Delta(K+1)e^{-K}\right)^{mn} \quad (31)\end{aligned}$$

where the last equality holds since $\sum_l \phi_l = {\rm tr}(\bar{\mathbf{H}}\bar{\mathbf{H}}^\dagger) = mn$. Then, from (24), the RHS of (31) is equal to one, and (30) can be obtained. □

Note that the asymptotic SNR gap is irrelevant to the target rate R and the number of transmit and receive antennas, denoted by n_T and n_R, respectively, and increases exponentially with the Rician factor K.

Example 1: Computer simulation has been performed to check the accuracy of (30). The simulated channels are 2×2 MIMO channels with $K = 0, 1, 2$, and $R = 10$. The channel matrix in (3) is generated 10^9 times for each K, and $P_{\rm out}(R,\rho)$ in (7) is estimated. The results are shown in Fig. 1. As expected, for high SNR, the slopes of the outage curves look identical to one another. When the outage probability is less than 10^{-6} [$P_o < 10^{-6}$ in (24)], the SNR gaps for $K=1$ and 2 are 1.3 and 3.5 dB, respectively, while the theoretical asymptotic SNR gaps in (30) are 1.33 dB ($K=1$) and 3.91 dB ($K=2$). These results indicate that the analytical SNR gaps match well with the simulation results.

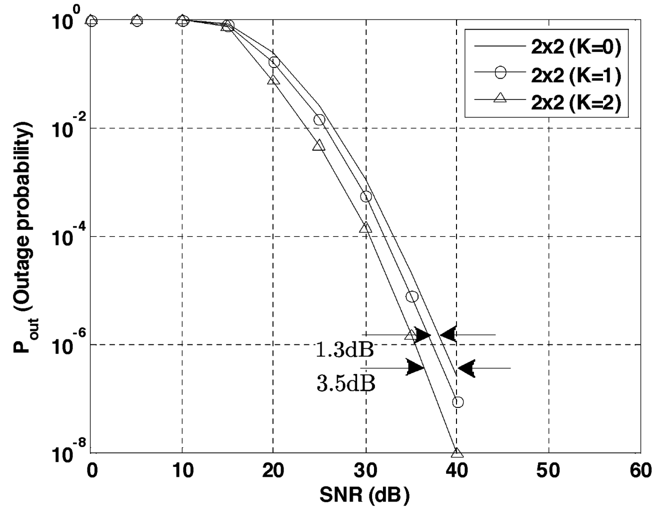

Fig. 1. Outage probabilities and asymptotic SNR gaps for $R=10, K=0,1,$ and 2.

IV. Finite SNR Analysis for Rician MISO and SIMO Channels

Finite SNR multiplexing and diversity gains are defined, and a closed-form expression for the outage probability is derived for Rician MISO and SIMO channels. Then, the slope of the outage probability is examined in a finite SNR regime and finite SNR DMT characteristics are analyzed.

A. Finite SNR Multiplexing and Diversity Gains

The finite SNR multiplexing gain $r_f(R,\rho,K)$ of a MIMO system with the target rate R and SNR ρ, operating over a Rician channel with factor K, is defined by the slope of the plot $R(\rho)$ versus $\log\rho$:

$$r_f(R,\rho,K) = \frac{\partial R(\rho)}{\partial\log\rho}. \quad (32)$$

Similarly, the finite SNR diversity gain is defined by the negative slope of the plot $\log P_{\rm out}(R,\rho,K)$ versus $\log\rho$ [25][2]:

$$d_f(R,\rho,K) = -\frac{\partial\log P_{\rm out}(R,\rho,K)}{\partial\log\rho}. \quad (33)$$

The consistency of these definitions with the original definitions of multiplexing and diversity gains in (5) is shown below.[3]

Lemma 4:

$$\lim_{\rho\to\infty} r_f = \lim_{\rho\to\infty}\frac{R(\rho)}{\log\rho} \quad (34)$$

[2]To simplify notations, $r_f(R,\rho,K)$ and $d_f(R,\rho,K)$ will be written as r_f and d_f, respectively, if dropping (R,ρ,K) does not cause any confusion.

[3]In [25], the finite SNR multiplexing gain is defined as $r(R,\rho,K) = R(\rho)/\log(1+g\rho)$, where g is a constant called the *array gain*. This definition also converges to the multiplexing gain r in (5a) as ρ goes to infinity. We use the definition in (32) because if $R(\rho)$ is linear with respect to $\log\rho$, then $R(\rho)/\log\rho = r_f$ for any ρ.

and

$$\lim_{\rho\to\infty} d_f = -\lim_{\rho\to\infty}\frac{\log P_e(\rho)}{\log\rho}. \tag{35}$$

Proof: $\lim_{\rho\to\infty} r_f = \lim_{\rho\to\infty} \partial R(\rho)/\partial\log\rho = \lim_{\rho\to\infty} R(\rho)/\log\rho$, where the second equality comes from the l'Hôspital's rule. Equation (35) can be obtained in a similar manner. □

Therefore, the finite SNR multiplexing and diversity gains can be viewed as extensions of the original definitions. The finite SNR diversity gain in (33) can be rewritten as

$$\begin{aligned} d_f &= -\lim_{\Delta(\log\rho)\to 0}\frac{\log P_{\rm out}(R_1,\rho_1,K)-\log P_{\rm out}(R,\rho,K)}{\Delta(\log\rho)}\\ &= \lim_{\Delta(\log\rho)\to 0}\Bigg[\frac{\log P_{\rm out}(R,\rho,K)}{\Delta(\log\rho)}\\ &\quad -\frac{\log P_{\rm out}(R+r_f\Delta(\log\rho), 2^{\log\rho+\Delta(\log\rho)},K)}{\Delta(\log\rho)}\Bigg] \end{aligned} \tag{36}$$

where $\Delta(\log\rho)=\log\rho_1-\log\rho$ for some SNR ρ_1 in the neighborhood of ρ ($\rho_1 = 2^{\log\rho+\Delta(\log\rho)} > \rho$) and $R_1 = R + r_f\Delta(\log\rho)$. Here, $r_f\Delta(\log\rho)$ denotes the additional rate that can be achieved by increasing the SNR from ρ to ρ_1 ($\Delta(\log\rho)$ is a nonnegative constant). Note in (36) that d_f is represented explicitly in terms of r_f. This fact is useful for deriving the finite SNR DMT characteristic.

B. Outage Probability for MISO/SIMO Channels With Finite SNR

When $m=1$, which corresponds to either a MISO ($n_T\times 1$) or a SIMO ($1\times n_R$) system, the outage probability in (7) is reduced to

$$\begin{aligned} P_{\rm out}(R,\rho,K) &= \Pr\left\{\log\left(1+\frac{\rho}{n_T}\sum_{l=1}^{n}|h_l|^2\right)<R\right\} && (37a)\\ &= \Pr\left\{\sum_{l=1}^{n}|h_l|^2<\frac{n_T(2^R-1)}{\rho}\right\} && (37b)\\ &= F_U(U<u(R,\rho)) && (37c)\end{aligned}$$

where $n=\max\{n_T,n_R\}$, h_l is the lth element of the channel vector, $U=\sum_l |h_l|^2$, $u(R,\rho)=n_T(2^R-1)/\rho$, and $F_U(\cdot)$ denotes the cumulative density function (cdf) of U. In this case, a closed-form expression for the outage probability can be obtained.

Lemma 5: For MISO/SIMO systems $(m=1)$ over a Rician channel, the outage probability in (37) can be represented as

$$P_{\rm out}(y)=1-Q_n\left(\sqrt{2Kn},\sqrt{\frac{K}{2n}}y\right) \tag{38}$$

where $y=\sqrt{(K+1)u/K}$, $u=n_T(2^R-1)/\rho$, and $Q_n(\cdot,\cdot)$ is the generalized Marcum Q-function [34] (to simplify notations, $u(R,\rho)$ is written as u). Furthermore, the pdf of y is given by

$$f_Y(y)=K\left(\frac{y}{2n}\right)^n e^{-Kn-Ky^2/4n}I_{n-1}(Ky) \tag{39}$$

where $I_{n-1}(\cdot)$ is the modified Bessel function of the first kind.

Proof: The sum $\sum_l |h_l|^2$ is a complex noncentral chi-squared random variable with n degrees of freedom and the noncentrality parameter Kn. Its pdf is given by [39]

$$\begin{aligned} f_U(u) &= (K+1)\left(\frac{(K+1)u}{Kn}\right)^{(n-1)/2} e^{-Kn-(K+1)u}\\ &\quad I_{n-1}\left(2\sqrt{K(K+1)nu}\right). \end{aligned} \tag{40}$$

Using this pdf, the following outage probability is derived in [40]:

$$P_{\rm out}(u,K)=1-Q_n\left(\sqrt{2Kn},\sqrt{2(K+1)u}\right). \tag{41}$$

The expressions in (38) and (39) come from (41) and (40), respectively, through changing the variables. □

An asymptotic outage probability for large K is presented in the following lemma.

Lemma 6: When the Rician factor K is large, the outage probability shown in (38) can be represented as

$$P_{\rm out}(y)=\begin{cases}\bar{P}_o(n,y), & \text{if } 2n>y\ge 0\\ 1+\bar{P}_o(n,y), & \text{if } y>2n\ge 0\end{cases} \tag{42}$$

where $n=\max\{n_T,n_R\}$, y is defined in Lemma 5, and

$$\begin{aligned}\bar{P}_o(n,y) &= \frac{e^{-Kn-Ky^2/4n+Ky}y^{n-1}}{\sqrt{2\pi Ky}2^{n+2}K(2n-y)^3 n^{n-1}}\\ &\quad\Big((4n^2-4ny+y^2)8Ky-16n^4+4n^2\\ &\quad +(16n^3-16n^2-12n)y\\ &\quad +(-4n^2+8n-3)y^2+O\left(\frac{1}{K}\right)\Big). \end{aligned} \tag{43}$$

The proof of this lemma is presented in Appendix II.

Example 2: Computer simulation has been performed to confirm the results in Lemma 5. The simulated channels are 2×1 MISO channel with $K=3,10,13$, and $R=12$. The channel matrix in (3) is generated 10^{10} times for each K, and $P_{\rm out}(R,\rho,K)$ is estimated. The results, shown in Fig. 2, indicate that the theoretical outage curves match well with the simulation results. (The simulation curves are shown for $P_{\rm out}(R,\rho,K)<10^{-9}$, while the theoretical curves are extended up to $P_{\rm out}(R,\rho,K)<10^{-16}$.)

An interesting observation that can be made in Fig. 2 is that the slope of the outage curves varies depending on the SNR ρ. When $K>0$, the maximum of the negative slope occurs at some finite SNR, and finding such an SNR value would be of

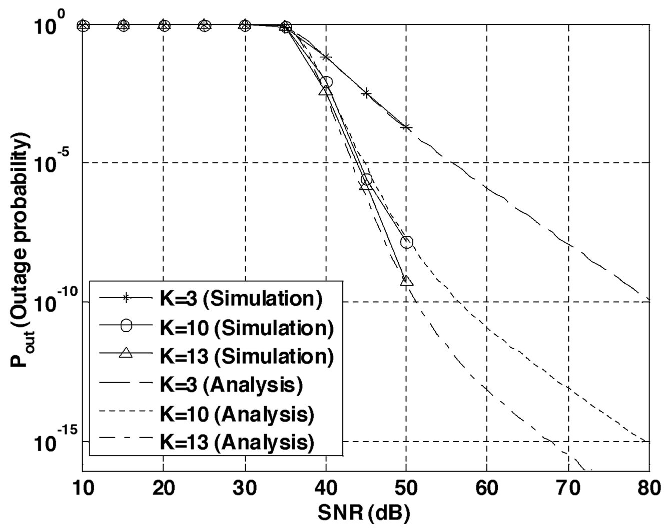

Fig. 2. Outage probability curves for 2×1 MISO channels with $K = 3, 10, 13$, and $R = 12$.

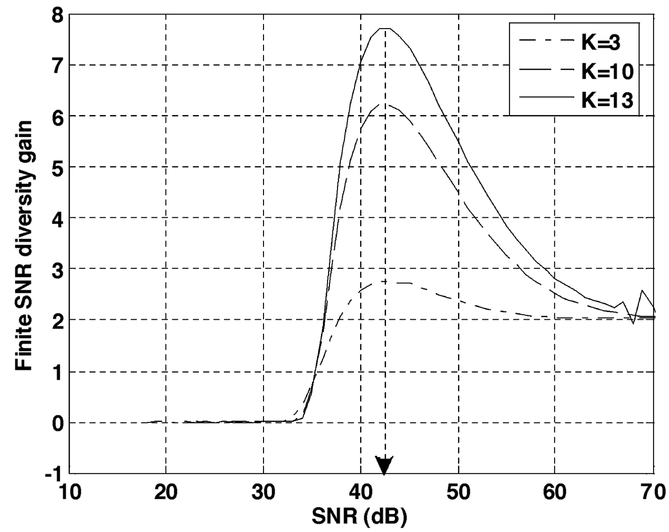

Fig. 3. Finite SNR diversity gains corresponding to the outage probability curves in Fig. 2.

practical importance. This issue will be addressed in the next section.

C. Finite SNR Maximum Diversity Gain for a Fixed Rate

The objective of this section is to find the maximum of $d_f(R, \rho, K)$ when R is fixed and K is large. It will be shown that the maximum diversity gain, denoted by $d^*_{f,\max}$, can be achieved at some finite SNR, say ρ^*, i.e., $d^*_{f,\max} \triangleq \max_{\rho} d_f(R, \rho, K) = d_f(R, \rho^*, K)$.

Theorem 5: For Rician MISO/SIMO channels with factor K, the finite SNR maximum diversity gain $d^*_{f,\max}$ is given by

$$d^*_{f,\max} = \frac{n}{4}K + \frac{1+2n}{4} + O\left(\frac{1}{K}\right) \tag{44}$$

and the maximum is achieved at

$$\rho^* = \frac{4n_T(2^R - 1)}{n} + O\left(\frac{1}{K}\right). \tag{45}$$

This theorem can be proved using Lemma 6 (see Appendix III for the proof). It is interesting to note that for large K, the maximum diversity gain $d^*_{f,\max}$ tends to increase linearly with K and is independent of the target rate R. The optimal SNR ρ^* is almost independent of K.

Example 3: To confirm the expression for ρ^* in (45), we have numerically obtained the negative slopes of the outage probability curves in Fig. 2. (The slopes are evaluated for each 1-dB interval of SNR, assuming that the slope remains constant in the interval.) The results are shown in Fig. 3. The maximum diversity gains are achieved at SNR $= 42$ dB irrespective of K, which is almost identical to $10\log_{10}(4n_T(2^R - 1)/n) \approx 42.14$ in (45).

Example 4: To confirm the expression of $d^*_{f,\max}$ in (44), $d^*_{f,\max}$ is obtained using the outage expression in (38). The 2×1 and 4×1 MISO channels with $R = 6$ and 12 are considered. The Rician factor K is varied from 2 to 15 and the negative slope of the outage probability is evaluated at SNR $= 24$ and 42 dB, which are the optimal SNRs when $R = 6$ and 12, respectively. The results are shown in Fig. 4. For comparison, the analytical values given by $(nK + 1 + 2n)/4$ in (44) are also shown. As predicted by the analysis, $\{d^*_{f,\max}\}$ for $R = 6$ and 12, which are numerically obtained from (38), are almost identical to each other and their $\{d^*_{f,\max}\}$ curves overlap. It is seen that the numerical and analytical results match well even for relatively small K values.

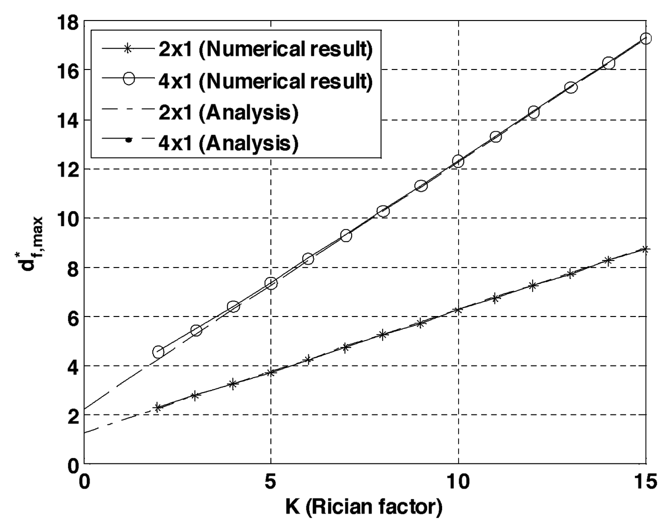

Fig. 4. Finite SNR maximum diversity gains $d^*_{f,\max}$ for 2×1 and 4×1 MISO channels with $R = 6$ and 12.

A Rician channel converges to an AWGN channel as K goes to infinity. Therefore, it is worthwhile to compare the results in Theorem 5 with the corresponding characteristics of AWGN channels. The outage probability of an AWGN channel is represented as

$$P_{\text{out}}(R, \rho) = \begin{cases} 1, & \text{if } \rho < \rho_{th} \\ 0, & \text{otherwise} \end{cases} \tag{46}$$

where ρ_{th} is a threshold value given by

$$\rho_{th} = \frac{n_T\left(2^R - 1\right)}{n}. \tag{47}$$

[This can be obtained directly from (37b).] Equation (46) indicates that the maximum diversity gain of an AWGN channel is infinity. This coincides with the result in (44), showing that $d^*_{f,\max}$ approaches infinity as K goes to infinity. However, ρ_{th} in (47) is different from the first term of the RHS of (45). Specifically

$$\lim_{K\to\infty} \frac{\rho^*}{\rho_{th}} = 4. \tag{48}$$

Therefore, ρ^* is asymptotically 6 dB higher than ρ_{th}.

D. Finite SNR DMT for MISO/SIMO Channels

In this section, we will derive the DMT characteristic for MISO/SIMO channels over a finite SNR regime. It will be shown that, as in the case of conventional DMT for MISO/SIMO channels over infinite SNR regime, the diversity gain can be represented as a linear function of the multiplexing gain.

Let $r_{f,\max}$ and $d_{f,\max}$ denote the finite SNR maximum multiplexing and diversity gains, respectively, which are defined as the multiplexing gain r_f when $d_f = 0$ and the diversity gain d_f when $r_f = 0$. The relation between d_f and r_f is presented in the following theorem.

Theorem 6: For MISO/SIMO Rician channels, the finite SNR DMT is given by

$$d_f = d_{f,\max} - \frac{d_{f,\max}}{r_{f,\max}} r_f \tag{49}$$

where $r_{f,\max} = 1 - 2^{-R}$.

Proof: We first obtain $r_{f,\max}$. From the definition of $r_{f,\max}$ and (36), we get

$$\lim_{\Delta(\log\rho)\to 0}\left[\frac{\log P_{\text{out}}(R,\rho,K)}{\Delta(\log\rho)} - \frac{\log P_{\text{out}}(R + r_{f,\max}\Delta(\log\rho), 2^{\log\rho+\Delta(\log\rho)}, K)}{\Delta(\log\rho)}\right] = 0.$$

Using the outage expression in (37c), this can be rewritten as

$$\lim_{\Delta(\log\rho)\to 0}\left[\frac{\log F_U(U<u(R,\rho))}{\Delta(\log\rho)} - \frac{\log F_U(U<u(R+r_{f,\max}\Delta(\log\rho), 2^{\log\rho+\Delta(\log\rho)}))}{\Delta(\log\rho)}\right] = 0. \tag{50}$$

The l'Hôspital's rule is applied to the left-hand side (LHS) of (50) by taking the derivatives of the numerator and the denominator with respect to $\Delta(\log\rho)$ and then setting $\Delta(\log\rho) = 0$. This results in

$$\frac{f_U(u(R + r_{f,\max}\Delta(\log\rho), 2^{\log\rho+\Delta(\log\rho)}))}{F(U < u(R + r_{f,\max}\Delta(\log\rho), 2^{\log\rho+\Delta(\log\rho)}))} \cdot \left.\frac{\partial u(R + r_{f,\max}\Delta(\log\rho), 2^{\log\rho+\Delta(\log\rho)})}{\partial\Delta(\log\rho)}\right|_{\Delta(\log\rho)=0} = 0$$

and

$$\frac{f_U(u(R,\rho))}{F(U<u(R,\rho))} \cdot \left.\frac{\partial u(R + r_{f,\max}\Delta(\log\rho), 2^{\log\rho+\Delta(\log\rho)})}{\partial\Delta(\log\rho)}\right|_{\Delta(\log\rho)=0} = 0. \tag{51}$$

Then,

$$\left.\frac{\partial u(R + r_{f,\max}\Delta(\log\rho), 2^{\log\rho+\Delta(\log\rho)})}{\partial\Delta(\log\rho)}\right|_{\Delta(\log\rho)=0} = \frac{n_T}{2^{\log\rho}}(r_{f,\max}2^R - 2^R + 1) = 0$$

and we get $r_{f,\max} = 1 - 2^{-R}$. Next, an expression for $d_{f,\max}$ is obtained,

$$\begin{aligned} d_{f,\max} &= d_f\Big|_{r_f=0} \\ &= -\lim_{\Delta(\log\rho)\to 0}\left[\frac{\log F_U(U<u(R,\rho))}{\Delta(\log\rho)} - \frac{\log F_U(U<u(R,2^{\log\rho+\Delta(\log\rho)}))}{\Delta(\log\rho)}\right] \\ &= \frac{-f_U(u(R,\rho))}{F_U(U<u(R,\rho))}\left.\frac{\partial u(R,2^{\log\rho+\Delta(\log\rho)})}{\partial\Delta(\log\rho)}\right|_{\Delta(\log\rho)=0} \\ &= \frac{f_U(u(R,\rho))u(R,\rho)}{F_U(U<u(R,\rho))}. \end{aligned} \tag{52}$$

Referring to (51), d_f can be represented as

$$d_f = \frac{f_U(u(R,\rho))}{F_U(U<u(R,\rho))} \cdot \left.\frac{\partial u(R + r_f\Delta(\log\rho), 2^{\log\rho+\Delta(\log\rho)})}{\partial\Delta(\log\rho)}\right|_{\Delta(\log\rho)=0} \tag{53}$$

and from (52) and the definition of $u(R,\rho)$ in (37), (53) becomes

$$\begin{aligned} d_f &= \frac{-d_{f,\max}}{u(R,\rho)}\frac{n_T}{\rho}\left(2^R(r_f - 1) + 1\right) \\ &= -\,d_{f,\max}\frac{2^R(r_f-1)+1}{2^R - 1} \\ &= \frac{d_{f,\max}}{r_{f,\max}}(r_{f,\max} - r_f). \end{aligned}$$

This completes the proof. □

The finite SNR DMT in (49) is consistent with the DMT for the infinite SNR case. For MISO/SIMO channels over infinite SNR regime, the DMT is expressed as (Theorems 1 and 2)

$$d^*(r) = n(1-r). \tag{54}$$

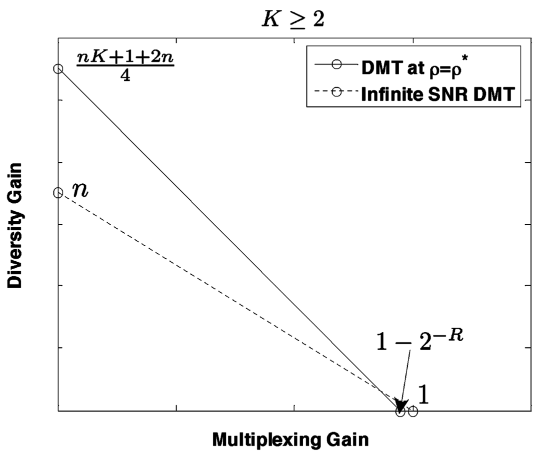

Fig. 5. DMT lines for Rician MISO/SIMO channels with $K \geq 2$, where $n = \max\{n_T, n_R\}$ and ρ^* is the SNR at which the maximum diversity gain is obtained for a given target rate R.

The DMT relation in (49) becomes identical to (54) as ρ becomes infinite because $\lim_{\rho\to\infty} r_{f,\max} = \lim_{R\to\infty}(1-2^{-R}) = 1$ and

$$\lim_{\rho\to\infty} d_{f,\max} = \lim_{\rho\to\infty} d_f|_{r_f=0} = -\lim_{\rho\to\infty} \left.\frac{P_e(\rho)}{\log\rho}\right|_{r=0} = n.$$

When $\rho = \rho^*$, the diversity gain $d_{f,\max}$ becomes $d^*_{f,\max}$ in (44) and for most Rician channels $d^*_{f,\max}$ is larger than n. For example, $d^*_{f,\max} \approx (nK+1+2n)/4$ for $K \geq 2$ (recall Example 4), and thus $d^*_{f,\max}$ tends to be greater than n for $K \geq 2$. The DMT line with $\rho = \rho^*$ is compared with the conventional DMT (infinite SNR) in Fig. 5. MISO/SIMO systems operating at the finite SNR ρ^* can enjoy considerably larger diversity gains for most values of the multiplexing gains than those with infinite SNR, and the loss in the maximum multiplexing gain 2^{-R} would not be severe if $R \geq 4$.

V. Conclusion

The DMT [15] and asymptotic outage expression for Rayleigh channels have been extended for Rician channels. The asymptotic SNR gap, representing the SNR difference between the outage curves of Rayleigh and Rician channels, was shown to increase exponentially with the Rician factor K. The DMT analysis for MISO/SIMO channels in a finite SNR regime resulted in the closed-form expressions for the maximum diversity gain and the optimal SNR at which the maximum is achieved. Finally, the finite SNR diversity gain was expressed as a linear function of the finite SNR multiplexing gain.

The finite SNR DMT in Section IV can be directly applied to the orthogonal space–time block code (OSTBC) [5] because an OSTBC can be viewed as a MISO channel with mn transmit antennas.

Further work in this area includes the characterization of DMT and outage performance for multiple-access MIMO channels and cooperative relay channels.

Appendix I
Proof of Lemma 2

It is found that the series expansion of the ${}_0F_1(\cdot,\cdot)$ scalar hypergeometric function yields [36]

$$ {}_0F_1(n-m+1, K(K+1)\phi_k\lambda_l) = \sum_{p=0}^{\infty} \frac{(K(K+1)\phi_k\lambda_l)^p}{p![n-m+1]_p} \tag{55}$$

where $[a]_b = (a+b-1)!/(a-1)!$. Let $\sum_{(k_1,\ldots,k_m)}$ denote the summation over all permutations $(k_1,\ldots,k_m)$ of $(1,\ldots,m)$. Then, from (55), the determinant can be written as

$$\begin{aligned}
&\det(\{{}_0F_1(n-m+1, K(K+1)\phi_k\lambda_l)\})\\
&= \sum_{(k_1,\ldots,k_m)} (-1)^{\mathrm{sign}(k_1,\ldots,k_m)} \cdot \prod_{l=1}^{m} {}_0F_1(n-m+1, K(K+1)\phi_{k_l}\lambda_l)\\
&= \sum_{(k_1,\ldots,k_m)} (-1)^{\mathrm{sign}(k_1,\ldots,k_m)} \cdot \prod_{l=1}^{m} \left\{ \sum_{p=0}^{\infty} \frac{(n-m)!(K(K+1)\phi_{k_l}\lambda_l)^p}{p!(n-m+p)!} \right\}\\
&= [(n-m)!]^m [K(K+1)]^{\bar{m}} \left(\prod_{k=1}^{m-1} \frac{1}{k!(n-m+k)!} \right)\\
&\quad \cdot \det\left(\begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_m \\ \vdots & \ddots & \vdots \\ \lambda_1^{m-1} & \cdots & \lambda_m^{m-1} \end{bmatrix} \right)\\
&\quad \cdot \det\left(\begin{bmatrix} 1 & \cdots & 1 \\ \phi_1 & \cdots & \phi_m \\ \vdots & \ddots & \vdots \\ \phi_1^{m-1} & \cdots & \phi_m^{m-1} \end{bmatrix} \right) (1+O(\lambda_m))\\
&= [(n-m)!]^m [K(K+1)]^{\bar{m}} \left(\prod_{k=1}^{m-1} \frac{1}{k!(n-m+k)!} \right)\\
&\quad \cdot \left[\prod_{k<l}^{m} (\lambda_k - \lambda_l) \prod_{k<l}^{m} (\phi_k - \phi_l) \right] (1+O(\lambda_m))
\end{aligned} \tag{56}$$

where $\mathrm{sign}(k_1,\ldots,k_m)$ is 0 or 1 depending on the permutation being even or odd, respectively. Therefore, (13) holds.

Appendix II
Proof of Lemma 6

The generalized Marcum Q-function in (38) can be represented in a series form as [41]

$$Q_n(a,b) = \begin{cases} 1 - e^{-(a^2+b^2)/2} \cdot \sum_{k=n}^{\infty} \left(\frac{b}{a}\right)^k I_k(ab), & \text{if } a > b \geq 0 \\ e^{-(a^2+b^2)/2} \cdot \sum_{k=1-n}^{\infty} \left(\frac{a}{b}\right)^k I_k(ab), & \text{if } b > a \geq 0 \end{cases} \tag{57}$$

and the outage probability in (38) can be rewritten as

$$P_{\text{out}}(y) = \begin{cases} e^{-Kn-Ky^2/4n} \\ \cdot \sum_{k=n}^{\infty} \left(\frac{y}{2n}\right)^k I_k(Ky), & \text{if } 2n > y \geq 0 \\ 1 - e^{-Kn-Ky^2/4n} \\ \cdot \sum_{k=1-n}^{\infty} \left(\frac{2n}{y}\right)^k I_k(Ky), & \text{if } y > 2n \geq 0. \end{cases} \tag{58}$$

We first consider the case where $2n > y \geq 0$. With the expression for the modified Bessel function of the first kind given by [36]

$$I_n(Ky) = \frac{e^{Ky}}{\sqrt{2\pi Ky}}\left(1+\frac{1-4n^2}{8Ky}+O\left(\frac{1}{K^2}\right)\right) \tag{59}$$

the outage probability in (58) is represented as

$$\begin{aligned}\bar{P}_o(n,y) &= e^{-Kn-Ky^2/4n}\sum_{k=n}^{\infty}\left(\frac{y}{2n}\right)^k I_k(Ky) \\ &= \frac{e^{-Kn-Ky^2/4n+Ky}}{\sqrt{2\pi Ky}} \\ &\quad \cdot\left(\sum_{k=n}^{\infty}\left(\frac{y}{2n}\right)^k\left(1+\frac{1-4k^2}{8Ky}\right)+O\left(\frac{1}{K^2}\right)\right) \\ &= \frac{e^{-Kn-Ky^2/4n+Ky}y^{n-1}}{\sqrt{2\pi Ky}2^{n+2}K(2n-y)^3n^{n-1}} \\ &\quad \cdot\Big((4n^2-4ny+y^2)8Ky-16n^4+4n^2 \\ &\quad +(16n^3-16n^2-12n)y \\ &\quad +(-4n^2+8n-3)y^2+O\left(\frac{1}{K}\right)\Big)\end{aligned} \tag{60}$$

which is the result in (43). When $y > 2n \geq 0$, following the approach similar to the first case, we obtain

$$P_{\text{out}}(y) = 1 + \bar{P}_o(n,y). \tag{61}$$

Appendix III
Proof of Theorem 5

This can be proved using the outage expression in Lemma 6. For simplicity, let $\gamma = \log\rho$ and $q(y) = \log\bar{P}_o(n,y)$, where $\bar{P}_o(n,y)$ is defined in (43). We first consider the case where $2n > y \geq 0$. Then, $P_{\text{out}}(R,\rho,K) = \bar{P}_o(n,y)$ and the maximum diversity gain is achieved at y satisfying

$$\frac{\partial^2}{\partial\gamma^2}q(y) = 0. \tag{62}$$

After some calculation, the first derivative of $q(y)$ is written as

$$\begin{aligned}\frac{\partial}{\partial\gamma}q(y) &= \frac{f_Y(y)}{(\ln 2)\bar{P}_o(n,y)}\left(\frac{\partial y}{\partial\gamma}\right) \\ &= -\frac{y}{2}\frac{f_Y(y)}{\bar{P}_o(n,y)}\end{aligned} \tag{63}$$

where $f_Y(y)$ is given in (39). Using the fact that $\partial\bar{P}_o(n,y)/\partial y = f_Y(y)$, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial\gamma^2}q(y) &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial\gamma}q(y)\right)\cdot\left(\frac{\partial y}{\partial\gamma}\right) \\ &= \frac{(\ln 2)y}{4[\bar{P}_o(n,y)]^2}\Big(\frac{df_Y(y)}{dy}\bar{P}_o(n,y)y \\ &\quad + f_Y(y)\bar{P}_o(n,y) - [f_Y(y)]^2y\Big).\end{aligned} \tag{64}$$

To evaluate the point of inflection satisfying (62), denoted by y^*, we need the following expressions for $f_Y(y)$ and $\partial f_Y(y)/\partial y$:

$$\begin{aligned}f_Y(y) &= K\left(\frac{y}{2n}\right)^n e^{-Kn-Ky^2/4n+Ky}\frac{1}{\sqrt{2\pi Ky}} \\ &\quad \cdot\left(1+\frac{1-4(n-1)^2}{8Ky}+O\left(\frac{1}{K^2}\right)\right)\end{aligned} \tag{65}$$

and

$$\begin{aligned}\frac{\partial f_Y(y)}{\partial y} &= \frac{K}{y}\left(\frac{y}{2n}\right)^n e^{-Kn-Ky^2/4n+Ky}\frac{1}{\sqrt{2\pi Ky}} \\ &\quad \cdot\left(\left(1-\frac{y}{2n}\right)Ky-\frac{4n^2-16n+7}{8}\right. \\ &\quad \left.+\frac{4n^2-8n+3}{16n}y+O\left(\frac{1}{K}\right)\right)\end{aligned} \tag{66}$$

where (65) and (66) are derived using the expressions for the modified Bessel functions of the first kind presented as follows [36]:

$$I_n(Ky) = \frac{e^{Ky}}{\sqrt{2\pi Ky}}\left(1+\frac{1-4n^2}{8Ky}+O\left(\frac{1}{K^2}\right)\right)$$

and

$$\frac{\partial}{\partial y}I_n(y) = \frac{1}{2}\left(I_{n-1}(y)+I_{n+1}(y)\right).$$

Using (43), (65), and (66) in (63) and (64), we obtain

$$\frac{\partial}{\partial\gamma}q(y) = \frac{r_0(y)}{r_1(y)}$$

and

$$\frac{\partial^2}{\partial\gamma^2}q(y) = \frac{(\ln 2)Ky}{4[\bar{P}_o(n,y)]^2}\left(\frac{4y^2(n-y)}{n(-2n+y)^2}+O\left(\frac{1}{K}\right)\right) \tag{67}$$

where

$$r_0(y) = Ky(2n-y)^3\left(8Ky+1-4(n-1)^2+O\left(\frac{1}{K}\right)\right)$$

and

$$\begin{aligned}r_1(y) = &-4n\Big((4n^2-4ny+y^2)8Ky-16n^4+4n^2 \\ &+(16n^3-16n^2-12n)y \\ &+(-4n^2+8n-3)y^2+O\left(\frac{1}{K}\right)\Big).\end{aligned}$$

The result in (67) indicates that

$$y^* = n + O\left(\frac{1}{K}\right). \quad (68)$$

From the definitions of y and u in Lemma 3, the SNR is written as

$$\rho = \frac{4(K+1)n \cdot n_T(2^R - 1)}{Ky^2}. \quad (69)$$

When $y = y^*$, the SNR in (69) becomes the desired expression in (45). Now

$$\begin{aligned} d^*_{f,\max} &= -\left.\frac{\partial q(y)}{\partial \gamma}\right|_{y=y^*} \\ &= -\frac{r_0(y^*)}{r_1(y^*)} \\ &= \frac{n}{4}K + \frac{1+2n}{4} + O\left(\frac{1}{K}\right) \end{aligned}$$

which is the result in (44). When $y > 2n \geq 0$, $q(y)$ in (62) is given by $q(y) = \log(1 + \bar{P}_o(n, y))$. In this case, following the approach similar to the first case, we can show that the point of inflection y^* violates the constraint $y > 2n$. Therefore, the maximum diversity gain cannot be achieved for this case.

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